Cubic Helices in Minkowski Space

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Abstract

We discuss space–like and light–like polynomial cubic curves in Minkowski (or pseudo–Euclidean) space $\mathbb{R}^{2,1}$ with the property that the Minkowski–length of the first derivative vector (or hodograph) is the square of a polynomial. These curves, which are called Minkowski Pythagorean hodograph (MPH) curves, generalize a similar notion from the Euclidean space (see Farouki, 2002). They can be used to represent the medial axis transform (MAT) of planar domains, where they lead to domains whose boundaries are rational curves. We show that any MPH cubic (including the case of light–like tangents) is a cubic helix in Minkowski space. Based on this result and on certain properties of tangent indicatrices of MPH curves, we classify the system of planar and spatial MPH cubics.

Key words: Helix, Minkowski space, Minkowski Pythagorean hodograph curve.

1 Introduction

Cubic curves with constant slope in Euclidean space have thoroughly been investigated by Wunderlich (1973). For any given slope α, there exists exactly one cubic curve in three–dimensional Euclidean space – which is called the cubic helix – for which the ratio of curvature to torsion equals α. Its normal form for $\alpha = \pi/4$ is given by

$$\mathbf{c}(t) = (3t^2, t - 3t^3, t + 3t^3)^\top. \tag{1}$$

Cubic helices for other slopes α are obtained by a uniform scaling of the z–coordinate. According to Wagner and Ravani (1997), cubic helices are the only cubics which are

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equipped with a rational Frenet–Serret motion. More precisely, the unit tangent, normal and binormal of the curve can be described by rational functions.

Pythagorean hodograph (PH) curves in Euclidean space were introduced by Farouki and Sakkalis (1990). While the only planar PH cubic is the so–called Tschirnhausen cubic, Farouki and Sakkalis (1994) proved later that spatial PH cubics are helices, i.e. curves of constant slope. A classification of PH cubics in Euclidean space can be obtained by combining these results: any PH cubic can be constructed as a helix with any given slope "over" the Tschirnhausen cubic.

Later, this notion was generalized to Minkowski (pseudo–Euclidean) space. As observed by Moon (1999) and by Choi et al. (1999), Minkowski Pythagorean hodograph (MPH) curves are very well suited for representing the so–called medial axis transforms (MAT) of planar domains.

Recall that the MAT of a planar domain is the closure of the set containing all points (x, y, r), where the circle with center (x, y) and radius r touches the boundary in at least two points and is fully contained within the domain. When the MAT is an MPH curve, the boundary curves of the associated planar domain admit rational parameterizations. Moreover, rational parameterizations of their offsets exist too, since the offsetting operations correspond to a translation in the direction of the time axis, which clearly preserves the MPH property.

These observations served to motivate constructions for MPH curves. Interpolation by MPH quartics was studied by Kim and Ahn (2003). Recently, it was shown that any space–like MAT can approximately be converted into a G^1 cubic MPH spline curve (Kosinka and Jüttler, 200x).

This paper analyzes the geometric properties of MPH cubics. As the main result, it is shown that these curves are again helices and can be classified, similarly to the Euclidean case.

The remainder of this paper is organized as follows. Section 2 summarizes some basic notions and facts concerning three–dimensional Minkowski geometry, MPH curves, and the differential geometry of curves in Minkowski space. Section 3 recalls some properties of helices in Euclidean space and it discusses helices in Minkowski space. Section 5 presents a classification of planar MPH cubics. Based on these results and using the so–called tangent indicatrix of a space–like curve we give a complete classification of spatial MPH cubics. Finally, we conclude the paper.

2 Preliminaries

In this section we summarize some basic concepts and results concerning Minkowski space, MPH curves and differential geometry of curves in Minkowski space.

2.1 Minkowski space

The three–dimensional Minkowski space $\mathbb{R}^{2,1}$ is a real linear space with an indefinite inner product given by the matrix $G = \mathrm{diag}(1, 1, -1)$. The inner product of two vectors $\mathbf{u} = (u_1, u_2, u_3)^\top$, $\mathbf{v} = (v_1, v_2, v_3)^\top$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$ is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top G \mathbf{v} = u_1 v_1 + u_2 v_2 - u_3 v_3. \tag{2}$$

The three axes spanned by the vectors $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$ will be denoted as the x–, y– and r–axis, respectively.

Since the quadratic form defined by G is not positive definite as in the Euclidean case, the square norm of $\mathbf{u}$ defined by $||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ may be positive, negative or zero. Motivated by the theory of relativity one distinguishes three so-called 'causal characters' of vectors. A vector $\mathbf{u}$ is said to be space–like if $||\mathbf{u}||^2 > 0$, time–like if $||\mathbf{u}||^2 < 0$, and light–like (or isotropic) if $||\mathbf{u}||^2 = 0$.

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$ are said to be orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. The cross–product in the Minkowski space can be defined analogously to the Euclidean case as

$$\mathbf{w} = \mathbf{u} \bowtie \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, -u_1 v_2 + u_2 v_1)^\top. \tag{3}$$

Clearly, $\langle \mathbf{u}, \mathbf{u} \bowtie \mathbf{v} \rangle = 0$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$.

A vector $\mathbf{u} \in \mathbb{R}^{2,1}$ is called a unit vector if $||\mathbf{u}||^2 = \pm 1$. The hyperboloid of one sheet given by $x^2 + y^2 - r^2 = 1$ spanned by the endpoints of all unit space–like vectors will be called the unit hyperboloid $\mathcal{H}$.

Let u, v and w be three vectors in $\mathbb{R}^{2,1}$. A scalar triple product of u, v and w is defined as

$$[u, v, w] = \langle u, v \bowtie w \rangle. \tag{4}$$

The scalar triple product in Minkowski space is the same as in Euclidean space, since the sign change in Minkowski inner and cross product cancels out. Therefore $[u, v, w] = \det(u, v, w)$.

A plane in Minkowski space is called space–, time– or light–like if the restriction of the quadratic form defined by G on this plane is positive definite, indefinite nondegenerate or degenerate, respectively. The type of a plane ρ can be characterized by the Euclidean angle α included between ρ and the xy plane. For light–like planes, $\alpha = \frac{\pi}{4}$.

2.2 Lorentz transforms

A linear transform $L : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ is called a Lorentz transform if it maintains the Minkowski inner product, i.e. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle L\mathbf{u}, L\mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$. The group of all Lorentz transforms $\mathcal{L} = O(2, 1)$ is called the Lorentz group.

Let $K = (k_{i,j})_{i,j=1,2,3}$ be a Lorentz transform. Then the column vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ satisfy $\langle \mathbf{k}_i, \mathbf{k}_j \rangle = G_{i,j}$, $i, j \in \{1, 2, 3\}$, i.e. they form an orthonormal basis of $\mathbb{R}^{2,1}$.

From $\langle \mathbf{k}_3, \mathbf{k}_3 \rangle = G_{3,3} = -1$ one obtains $k_{33}^2 \geq 1$. A transform K is said to be orthochronous if $k_{33} \geq 1$. The determinant of any Lorentz transform K equals to ± 1, and special ones are characterized by $\det(K) = 1$.

The Lorentz group $\mathcal{L}$ consists of four components. The special orthochronous Lorentz transforms form a subgroup $SO_+(2, 1)$ of $\mathcal{L}$. The other components are $T_1 \cdot SO_+(2, 1)$, $T_2 \cdot SO_+(2, 1)$ and $T_1 \cdot T_2 \cdot SO_+(2, 1)$, where $T_1 = \mathrm{diag}(1, 1, -1)$ and $T_2 = \mathrm{diag}(1, -1, 1)$.

Let

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad H(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{pmatrix} \tag{5}$$

be a rotation of the spatial coordinates x, y, and a hyperbolic rotation with a hyperbolic angle β, respectively. Any special orthochronous Lorentz transform $L \in SO_+(2, 1)$ can be represented as $L = R(\alpha_1)H(\beta)R(\alpha_2)$.

The restriction of the hyperbolic rotation to the time–like yr–plane (i.e. to Minkowski space $\mathbb{R}^{1,1}$) is given by

$$h(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}. \tag{6}$$

2.3 MPH curves

A curve segment $\mathbf{c}(t) \in \mathbb{R}^{2,1}$, $t \in [a, b]$ is called space–, time– or light–like if its tangent vector $\mathbf{c}'(t)$, $t \in [a, b]$ is space–, time– or light–like, respectively.

Recall that a polynomial curve in Euclidean space is said to be a *Pythagorean hodograph* (PH) curve (cf. Farouki (2002)), if the norm of its first derivative vector (or "hodograph") is a (possibly piecewise) polynomial. Following Moon (1999), a *Minkowski Pythagorean hodograph* (MPH) curve is defined similarly, but with respect to the norm induced by the Minkowski inner product. More precisely, a polynomial curve $\mathbf{c} \in \mathbb{R}^{2,1}$, $\mathbf{c} = (x, y, r)^\top$ is called an MPH curve if

$$x'^2 + y'^2 - r'^2 = \sigma^2 \tag{7}$$

for some polynomial σ.

Remark 1 As observed by Moon (1999) and Choi et al. (1999), if the medial axis transform (MAT) of a planar domain is an MPH curve, then the coordinate functions of the corresponding boundary curves and their offsets are rational.

Remark 2 As an immediate consequence of the definition, the tangent vector $\mathbf{c}'(t)$ of an MPH curve cannot be time–like. Also, light–like tangent vectors $\mathbf{c}'(t)$ correspond to roots of the polynomial σ in (7).

2.4 Frenet formulas in Minkowski space

This section introduces several facts from the differential geometry of curves in Minkowski space, cf. Walrave (1995). We consider a curve segment $\mathbf{c}(t) \in \mathbb{R}^{2,1}$. In order to rule out straight line and inflections, we suppose that the first two derivative vectors $\mathbf{c}'(t)$ and $\mathbf{c}''(t)$ are linearly independent. More precisely, points with linearly dependent vectors $\mathbf{c}'(t)$ and $\mathbf{c}''(t)$ correspond to inflections in the sense of projective differential geometry, which will be excluded. We distinguish three different cases.

Case 1: Consider a space–like curve $\mathbf{c}(s) \in \mathbb{R}^{2,1}$, i.e. $||\mathbf{c}'(s)|| > 0$. We may assume that the curve is parameterized by its arc length, i.e. $||\mathbf{c}'(s)|| = 1$. Then we define a (space–like) unit tangent vector $\mathbf{T} = \mathbf{c}'(s)$ of $\mathbf{c}(s)$.

Subcase 1.1: If the vector $\mathbf{T}'$ is space–like or time–like on some parameter interval, the Frenet formulas take the form

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= - \langle \mathbf{N}, \mathbf{N} \rangle \kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' &= \tau \mathbf{N}. \end{aligned} \tag{8}$$

The unit vectors $\mathbf{N}$ and $\mathbf{B}$ are the unit normal and binormal vector, $\kappa > 0$ and τ are the Minkowski curvature and torsion of $\mathbf{c}(s)$, respectively. The three vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ form an orthonormal basis.

Subcase 1.2: The vector $\mathbf{T}'$ of a space–like curve may be light–like at an isolated point, or within an entire interval. The two cases will be called *Minkowski inflections* and *inflected segments*, respectively. The Frenet formulas of a space–like curve within an inflected segment take the form

$$\begin{aligned} \mathbf{T}' &= \mathbf{N}, \\ \mathbf{N}' &= \tau \mathbf{N}, \\ \mathbf{B}' &= -\mathbf{T} - \tau \mathbf{B}, \end{aligned} \tag{9}$$

where $\langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0$, $\langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 0$ and $\langle \mathbf{N}, \mathbf{B} \rangle = 1$. In this situation, the Minkowski curvature evaluates formally to $\kappa = 1$. This subcase covers curves lying in light–like planes.

Case 2: Consider a light–like curve $\mathbf{c}(s) \in \mathbb{R}^{2,1}$, i.e. $\langle \mathbf{c}'(s), \mathbf{c}'(s) \rangle = 0$. It follows that $\langle \mathbf{c}'(s), \mathbf{c}''(s) \rangle = 0$ and thus $\mathbf{c}''(s)$ lies in a light–like plane. Therefore $\mathbf{c}''(s)$ is space–like (light–like vector $\mathbf{c}''(s)$ leads to an inflection). We may assume that the curve is parameterized by its so called pseudo arc length, i.e. $||\mathbf{c}''(s)|| = 1$. Then we have

$$\begin{aligned} \mathbf{T}' &= \mathbf{N}, \\ \mathbf{N}' &= \tau \mathbf{T} - \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned} \tag{10}$$

where $\langle \mathbf{N}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0$, $\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 0$ and $\langle \mathbf{T}, \mathbf{B} \rangle = 1$. Again, the Minkowski curvature evaluates to $\kappa = 1$.

Case 3: Let us consider a time–like curve $\mathbf{c}(s) \in \mathbb{R}^{2,1}$ parameterized by its arc length, i.e. $||\mathbf{c}'(s)|| = -1$. Then we define a (time–like) unit tangent vector $\mathbf{T} = \mathbf{c}'(s)$. As $\langle \mathbf{T}, \mathbf{T}' \rangle = 0$, the vector $\mathbf{T}'$ lies in a space–like plane. Therefore, $\mathbf{T}'$ is always space–like. The Frenet formulas take the form

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= \kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned} \tag{11}$$

where $\mathbf{N}$ and $\mathbf{B}$ are the unit normal and binormal vector, $\kappa > 0$ and τ are the Minkowski curvature and torsion of $\mathbf{c}(s)$, respectively. The three vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ form an orthonormal basis.

Remark 3 In the remainder of the paper, the notion of inflection also includes Minkowski inflections.

The next result characterizes inflections.

Proposition 4 *Let $\mathbf{c}(t) \in \mathbb{R}^{2,1}$ be a space–like curve. Then $\mathbf{c}(t)$ has an inflection corresponding to $t \in I$ if and only if $||\mathbf{c}'(t) \bowtie \mathbf{c}''(t)||^2 = 0$ for $t \in I$. This includes the case of an isolated inflection point, where $I = \{t_0\}$.*

Proof: For the sake of brevity, we will omit the dependence on the parameter t. Let $\mathbf{T}$ and κ be the unit tangent vector and the curvature of $\mathbf{c}$. As in the Euclidean case, the Frenet formulas imply

$$||\mathbf{c}' \bowtie \mathbf{c}''||^2 = \kappa^2 ||\mathbf{c}'||^6 ||\mathbf{T} \bowtie \mathbf{T}'||^2. \tag{12}$$

Firstly, let $\mathbf{c}$ have an inflection. Then $\mathbf{T}'$ is a light–like vector. As $\langle \mathbf{T}, \mathbf{T} \rangle = 1$, by differentiating we obtain $\langle \mathbf{T}, \mathbf{T}' \rangle = 0$. One can easily check that $\langle \mathbf{T}, \mathbf{T}' \rangle = 0$ implies $||\mathbf{T} \bowtie \mathbf{T}'||^2 = 0$ (the geometric argument is that the vectors $\mathbf{T}$, $\mathbf{T}'$ define a light–like plane). Finally, (12) yields that $||\mathbf{c}' \bowtie \mathbf{c}''||^2 = 0$.

Secondly, let $||\mathbf{c}' \bowtie \mathbf{c}''||^2 = 0$. From (12) we get that $||\mathbf{T} \bowtie \mathbf{T}'||^2 = 0$ or $\kappa = 0$ has to hold. Again, since $\langle \mathbf{T}, \mathbf{T}' \rangle = 0$, we can conclude that the vector $\mathbf{T}'$ is light–like. $\square$

The formulas

$$\kappa(t) = \frac{\sqrt{|\langle \mathbf{c}'(t) \bowtie \mathbf{c}''(t), \mathbf{c}'(t) \bowtie \mathbf{c}''(t) \rangle|}}{||\mathbf{c}'(t)||^3}, \tag{13}$$

and

$$\tau(t) = \frac{[\mathbf{c}'(t), \mathbf{c}''(t), \mathbf{c}'''(t)]}{|\langle \mathbf{c}'(t) \bowtie \mathbf{c}''(t), \mathbf{c}'(t) \bowtie \mathbf{c}''(t) \rangle|}. \tag{14}$$

for the curvature and torsion of a space–like curve $\mathbf{c}(t)$ without inflections can be derived from Frenet formulas. The proof is similar to the Euclidean case.

2.5 Curves of zero curvature or torsion

Let us take a closer look at curves in Minkowski space with curvature or torsion identically equal to zero. One can verify that a curve $\mathbf{c}(t)$ in Euclidean or Minkowski space, whose curvature vanishes identically, is contained within a straight line.

Definition 5 *A curve in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ is called a spatial curve if and only if it does not lie in a plane.*

In the Euclidean space $\mathbb{R}^3$, a curve, which is not a straight line, is planar (non–spatial) if and only if its torsion is identically equal to zero. Analogously, one may ask for curves in $\mathbb{R}^{2,1}$ with vanishing torsion.

The answer to this question is not the same as in Euclidean case. In fact, $\tau \equiv 0$ is neither a necessary nor a sufficient condition for a curve to be planar in Minkowski space.

On the one hand, it can be shown that curves lying in light–like planes consist only of Minkowski inflections, hence they are planar without having vanishing torsion (The Minkowski curvature formally evaluates to 1, and the torsion plays the role of the curvature). These curves correspond to Subcase 1.2 of the Frenet formulas.

On the other hand, curves with vanishing torsion are described by the following result.

Proposition 6 (Walrave, 1995) *If a curve $\mathbf{c}(t) \in \mathbb{R}^{2,1}$ has vanishing torsion, then it is a planar curve or a curve similar to the so–called W–null–cubic*

$$\mathbf{w}(s) = \frac{1}{6\sqrt{2}}(6s - s^3, 3\sqrt{2}s^2, 6s + s^3)^\top. \tag{15}$$

Proof: (Sketch, see Walrave (1995) for details) Consider a curve $\mathbf{c}(s) \in \mathbb{R}^{2,1}$ such that $\tau \equiv 0$, $\kappa \neq 0$. When $\mathbf{c}(s)$ is space–like or time–like, one can easily verify that the third derivative $\mathbf{c}'''(s)$ of $\mathbf{c}(s)$ is a linear combination of $\mathbf{c}'(s)$ and $\mathbf{c}''(s)$, which implies that $\mathbf{c}(s)$ is a planar curve. When $\mathbf{c}(s)$ is light–like, $\tau \equiv 0$ yields $\mathbf{c}(s) = \frac{1}{6\sqrt{2}}(6s - s^3, 3\sqrt{2}s^2, 6s + s^3)^\top$.

Therefore, the only spatial curve in $\mathbb{R}^{2,1}$ (up to Minkowski similarities) with torsion identically equal to zero is the light–like curve (15), which we will refer to as the W–null–cubic. Note that $\mathbf{w}(s)$ is parameterized by its pseudo arc length. □

Remark 7 (1) The W–null–cubic is also an MPH curve, since any polynomial light–like curve is an MPH curve.

(2) Throughout this paper, *similar* refers to the Minkowski geometry, i.e., it means equal up to Lorentz transforms, translations, and scaling.

3 Helices in Minkowski space

We start with a brief summary of some basic results from Euclidean space.

A helix in $\mathbb{R}^3$ is a spatial curve for which the tangent makes a constant angle with a fixed line. Any such a line is called the axis of the helix. Lancret's theorem states that a necessary and sufficient condition for a spatial curve to be a helix in $\mathbb{R}^3$ is that the ratio of its curvature to torsion is constant. The proof of this theorem uses Frenet formulas and can be found in many textbooks on classical differential geometry, e.g. Kreyszig (1991).

In the Euclidean version of the Lancret's theorem the restriction to spatial curves rules out curves with vanishing torsion. However, as shown in Section 2.5, this is generally not the case in Minkowski space.

Definition 8 *A curve* $\mathbf{c}(t) \in \mathbb{R}^{2,1}$ *is called a helix if and only if there exists a constant vector* $\mathbf{v} \neq (0,0,0)^\top$ *such that* $\langle \mathbf{T}(t), \mathbf{v} \rangle$ *is constant, where* $\mathbf{T}(t)$ *is the unit tangent vector of* $\mathbf{c}(t)$*. Any line, which is parallel to the vector* $\mathbf{v}$*, is called an axis of the helix* $\mathbf{c}(t)$*.*

Proposition 9 (Lancret's theorem in $\mathbb{R}^{2,1}$) *A spatial curve* $\mathbf{c}(t) \in \mathbb{R}^{2,1}$ *is a helix if and only if* $\tau = \alpha\kappa$*, where* α *is a real constant and* κ*,* τ *are the Minkowski curvature and torsion of* $\mathbf{c}(t)$*.*

Proof: Recall that $\alpha = 0$ (i.e. $\tau \equiv 0$) corresponds to the W–null–cubic as shown in Proposition 6. From the Frenet formulas for light–like curves follows that the binormal vector $\mathbf{B}$ of the W–null–cubic is a constant vector and $\langle \mathbf{T}, \mathbf{B} \rangle = 1$. This implies that the W–null–cubic is a helix in $\mathbb{R}^{2,1}$.

Now, let us suppose that $\alpha \neq 0$. As the proof is analogous for all five different cases of curves and corresponding Frenet formulas, we provide the proof of Lancret's theorem for two of the cases only.

Let $\mathbf{c}'(t)$ be space–like and $\mathbf{c}''(t)$ not light–like and let $\mathbf{c}(t)$ be a helix in $\mathbb{R}^{2,1}$. Then there exists a constant vector $\mathbf{v} \neq (0,0,0)^\top$ such that $\langle \mathbf{T}, \mathbf{v} \rangle = \beta$, $\beta \in \mathbb{R}$. By differentiating this equation with respect to t and using Frenet formulas we obtain $\langle \mathbf{N}, \mathbf{v} \rangle = 0$ and thus $\mathbf{v} = a\mathbf{T} + b\mathbf{B}$, where $a, b \in \mathbb{R}$. Again, by differentiating we get $\mathbf{N}(a\kappa + b\tau) = 0$, which gives $\tau = -\frac{a}{b}\kappa$ ($b = 0$ implies that $\mathbf{c}(t)$ is a straight line).

Conversely, let $\tau = \alpha\kappa$. Then we choose the vector $\mathbf{v} = \mathbf{T} - \frac{1}{\alpha}\mathbf{B}$. By differentiating this equation with respect to t one obtains that $\mathbf{v}' = (0,0,0)^\top$, i.e. $\mathbf{v}$ is a constant vector. Moreover, $\langle \mathbf{T}, \mathbf{v} \rangle = \langle \mathbf{T}, \mathbf{T} - \frac{1}{\alpha}\mathbf{B} \rangle = 1$, which proves that $\mathbf{c}(t)$ is a helix in $\mathbb{R}^{2,1}$. □

Remark 10 For the remainder of the paper we restrict ourselves to space–like and light–like helices only. In order to avoid confusion, we will call these curves *SL–helices.*

Proposition 11 *Any polynomial SL–helix in Minkowski space is an MPH curve.*

Proof: Let $\mathbf{c}(t)$ be a polynomial light–like helix. Clearly, any polynomial light–like curve is an MPH curve.

Now, let $\mathbf{c}(t)$ be a polynomial space–like helix. Then there exists a constant vector $\mathbf{v}$ such that $\langle \mathbf{T}, \mathbf{v} \rangle = \alpha$, where α is a real constant. One can easily verify that $\alpha = 0$ leads to a contradiction with $\mathbf{c}(t)$ being a helix, since the torsion of $\mathbf{c}(t)$ would be identically equal to zero (cf. Proposition 6). Therefore $\alpha \neq 0$.

The unit tangent vector of $\mathbf{c}(t)$ can be obtained from $\mathbf{T} = \frac{\mathbf{c}'(t)}{\sqrt{\langle \mathbf{c}'(t), \mathbf{c}'(t) \rangle}}$. By substituting $\mathbf{T}$ in the first equation we obtain

$$\frac{\langle \mathbf{c}'(t), \mathbf{v} \rangle}{\alpha} = \sqrt{\langle \mathbf{c}'(t), \mathbf{c}'(t) \rangle}. \tag{16}$$

As the curve $\mathbf{c}(t)$ is a polynomial curve, the left–hand side of (16) is a polynomial. Consequently, the right–hand side of (16) is a polynomial as well and hence $\mathbf{c}(t)$ is an MPH curve. □

4 Spatial MPH cubics

In this section we discuss the connection between polynomial helices in Minkowski space and MPH curves.

4.1 Space–like MPH cubics

Proposition 12 *The ratio of curvature to torsion of a spatial space–like MPH cubic is constant. Consequently, spatial space–like MPH cubics are helices in* $\mathbb{R}^{2,1}$.

Proof: We will prove the proposition by a direct computation. Let $\mathbf{c}(t) = (x(t), y(t), r(t))^\top \in \mathbb{R}^{2,1}$ be a spatial space–like MPH cubic. Then there exist four linear polynomials (cf. Moon (1999))

$$\begin{array}{l} u(t) = u_0(1-t) + u_1 t,\ v(t) = v_0(1-t) + v_1 t, \\ p(t) = p_0(1-t) + p_1 t,\ q(t) = q_0(1-t) + q_1 t, \end{array} \tag{17}$$

such that

$$\begin{array}{l} x'(t) = u(t)^2 - v(t)^2 - p(t)^2 + q(t)^2, \\ y'(t) = -2(u(t)v(t) + p(t)q(t)), \\ r'(t) = 2(u(t)q(t) + v(t)p(t)), \\ \sigma(t) = u(t)^2 + v(t)^2 - p(t)^2 - q(t)^2. \end{array} \tag{18}$$

Since $\mathbf{c}(t)$ is a space–like curve, we may (without loss of generality) assume that $\mathbf{c}'(0) = (1, 0, 0)^\top$, which implies $u_0 = 0$, $p_0 = 0$ and $q_0^2 - v_0^2 = 1$. Expressing the curve $\mathbf{c}(t)$ in Bézier form and computing the following scalar triple product yield that $\mathbf{c}(s)$ is a planar curve if

$$[\mathbf{c}'(0), \mathbf{c}'(1), \mathbf{c}(1) - \mathbf{c}(0)] = (u_1 - p_1)(u_1 + p_1)(v_1 q_0 - q_1 v_0) = 0. \tag{19}$$

According to Proposition 4, $\mathbf{c}(t)$ has an inflection point if

$$(u_1 - p_1)(u_1 + p_1)\sigma(t) = 0. \tag{20}$$

One can observe from (19) and (20) that $\mathbf{c}(t)$ has an inflection if it has a light–like tangent (or it is a planar curve). Thus spatial space–like MPH cubics have no inflections.

Applying the formulas (13) and (14) to the curve $\mathbf{c}(t)$ gives that the curvature and torsion of $\mathbf{c}(t)$ are given by

$$\kappa(t) = \frac{2\sqrt{|(u_1 - p_1)(u_1 + p_1)|}}{\sigma^2(t)}, \quad \tau(t) = \frac{2(v_1 q_0 - q_1 v_0)}{\sigma^2(t)}. \tag{21}$$

Consequently, the ratio of $\kappa(t)$ to $\tau(t)$ does not depend on t. □

4.2 Light–like MPH cubics

Proposition 13 *Any spatial light–like cubic is similar to the W–null–cubic.*

Proof: Consider a spatial light–like polynomial curve $\mathbf{c}(t)$ of degree 3. Let $t = t(s)$ be a reparameterization of $\mathbf{c}(t)$ such that $\mathbf{c}(t(s))$ is parameterized by the pseudo arc length and let $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ be the tangent, normal and binormal vector and $\tau(s)$ the torsion of $\mathbf{c}(t)$. We denote by $\mathbf{c}'$ and $\dot{\mathbf{c}}$ the first derivative of $\mathbf{c}$ with respect to t and s, respectively. For the sake of brevity we omit the dependence on s. Then we have

$$\mathbf{T} = \mathbf{c}'(t)\frac{\mathrm{d}t}{\mathrm{d}s}. \tag{22}$$

Three consecutive differentiations of (22) with respect to s and simplifications using Frenet formulas yield

$$\begin{aligned} \mathbf{N} &= \mathbf{c}''(t)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \mathbf{c}'(t)\frac{\mathrm{d}^2t}{\mathrm{d}s^2}, \\ \tau\mathbf{T} - \mathbf{B} &= \mathbf{c}'''(t)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^3 + 3\mathbf{c}''(t)\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \mathbf{c}'(t)\frac{\mathrm{d}^3t}{\mathrm{d}s^3}, \\ \dot{\tau}\mathbf{c}'(t)\frac{\mathrm{d}t}{\mathrm{d}s} = \dot{\tau}\mathbf{T} &= \mathbf{c}''''(t)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^4 + a_3(s)\mathbf{c}'''(t) + a_2(s)\mathbf{c}''(t) + \mathbf{c}'(t)\frac{\mathrm{d}^4t}{\mathrm{d}s^4}, \end{aligned} \tag{23}$$

where $a_3(s) = \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \frac{\mathrm{d}^2t}{\mathrm{d}s^2} = (\dot{t})^2\ddot{t}$ and $a_2(s)$ is a function of s.

Consider the scalar triple product $[\mathbf{T}, \mathbf{N}, \tau\mathbf{T} - \mathbf{B}]$. Using (22) and (23) one obtains

$$[\mathbf{T}, \mathbf{N}, \tau\mathbf{T} - \mathbf{B}] = [\mathbf{T}, \mathbf{N}, -\mathbf{B}] = [\mathbf{c}'(t), \mathbf{c}''(t), \mathbf{c}'''(t)]\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^6. \tag{24}$$

Since the vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are linearly independent, the vectors $\mathbf{c}'(t)$, $\mathbf{c}''(t)$ and $\mathbf{c}'''(t)$ are linearly independent as well. Consequently, by comparing coefficients and due to the fact that $\mathbf{c}''''(t) = (0, 0, 0)^\top$, the third equation of (23) implies

$$a_3(s) = 0, \quad a_2(s) = 0, \quad \dot{\tau}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}^4t}{\mathrm{d}s^4}. \tag{25}$$

From $a_3(s) = 0$ one may conclude that $t = \alpha s + \beta$ and therefore the last equation of (25) implies that τ is constant.

Finally, we express the binormal vector $\mathbf{B}$ using the second equation of (23):

$$\begin{aligned} \mathbf{B} &= \tau\mathbf{T} - \alpha^3\mathbf{c}'''(t), \\ \dot{\mathbf{B}} &= \dot{\tau}\mathbf{T} + \tau\dot{\mathbf{T}} - \alpha^4\mathbf{c}''''(t) = \tau\mathbf{N}. \end{aligned} \tag{26}$$

On the other hand, from the Frenet formulas we have that $\dot{\mathbf{B}} = -\tau\mathbf{N}$. Therefore, the torsion τ is identically equal to zero. Proposition 6 concludes the proof. □

4.3 Summary

In this section we will summarize the previously obtained results (see the scheme in Fig. 1).

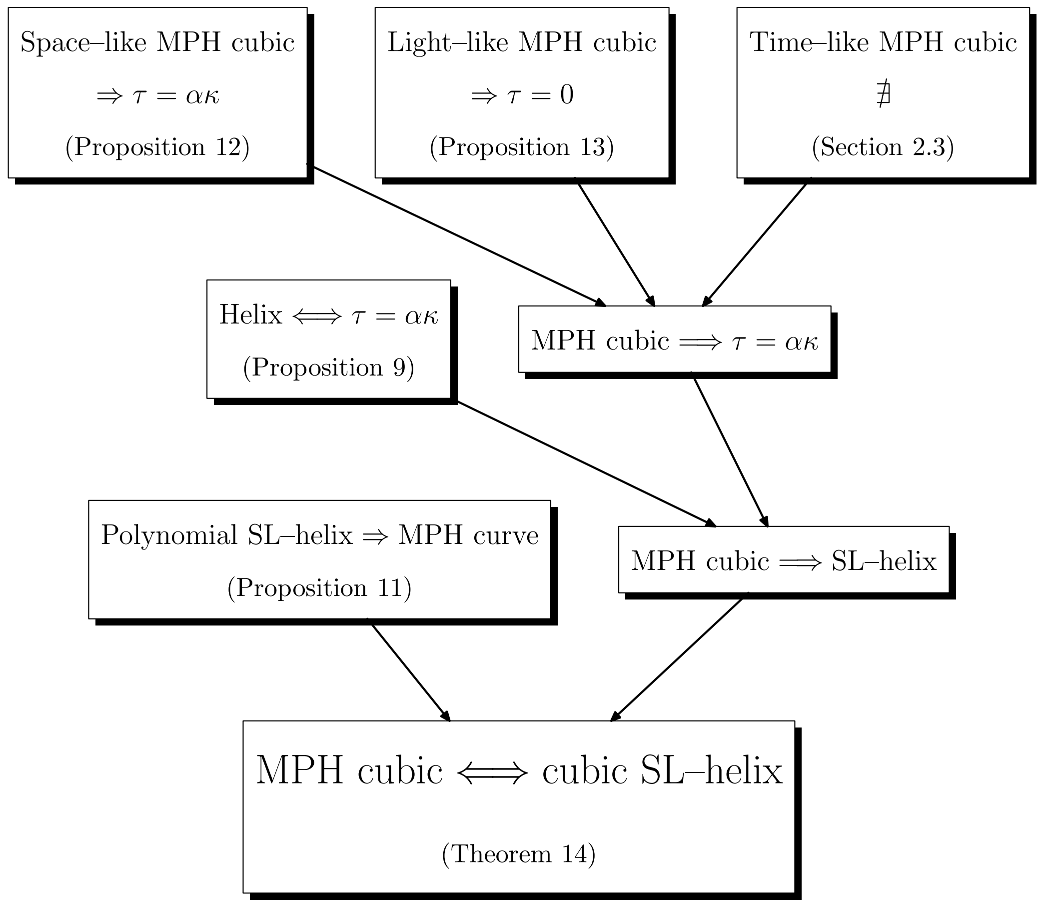

Fig. 1. Summary of obtained results concerning spatial MPH cubics.

Theorem 14 *A spatial curve in Minkowski space is an MPH cubic if and only if it is a space-like or light–like cubic helix.*

Proof: From Propositions 12 and 13 follows that any spatial MPH cubic satisfies the assumptions of the Lancret's theorem (Theorem 9) and therefore any such curve is a helix in Minkowski space. On the other hand, we have proved (cf. Proposition 11) that any polynomial SL–helix is an MPH curve. □

5 Classification of planar MPH cubics

In order to prepare the discussion of spatial helices, we present a classification of planar MPH cubics.

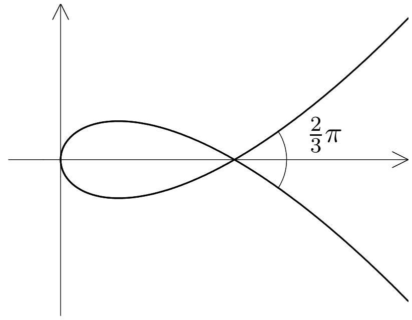

Fig. 2. Tschirnhausen cubic

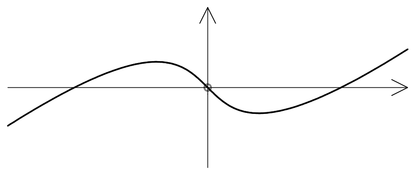

Fig. 3. MPH cubic in $\mathbb{R}^{1,1}$ with exactly one point with a light–like tangent (marked by the grey circle).

5.1 MPH cubics in space–like and light–like planes

A thorough discussion of planar MPH cubics in space–like planes was given in Farouki and Sakkalis (1990), since these curves are planar PH cubics. It turns out that any planar PH cubic (which is not a straight line) is similar to the so–called Tschirnhausen cubic (see Fig. 2) given by $\mathbf{T}(t) = (3t^2, t - 3t^3)^\top$.

Let us consider a polynomial planar curve $\mathbf{c}(t) = (x(t), y(t))^\top$. The MPH condition in a light–like plane degenerates to $x'^2(t) = \sigma^2(t)$ and hence any polynomial curve in a light–like plane is an MPH curve. Therefore the only case remaining to consider is the case of a time–like plane.

5.2 MPH cubics in time–like planes

An MPH curve $\mathbf{c}(t) = (x(t), y(t))^\top$ lying in a time–like plane is nothing else but a curve in Minkowski plane $\mathbb{R}^{1,1}$ whose hodograph satisfies $x'^2(t) - y'^2(t) = \sigma^2(t)$, where $\sigma(t)$ is a polynomial in t.

Proposition 15 *Any MPH cubic in Minkowski plane $\mathbb{R}^{1,1}$ with exactly one point with a light–like tangent is similar to the curve $\mathbf{q}_1(t) = (t^3 + 3t, t^3 - 3t)^\top$ (depicted in Fig. 3).*

Any MPH cubic in Minkowski plane $\mathbb{R}^{1,1}$ with exactly two different points with light–like tangents is similar (in Minkowski sense) to the curve $\mathbf{q}_2(t) = (t^3 + 3t, 3t^2)^\top$ (see Fig. 4).

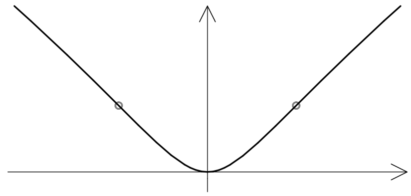

Fig. 4. MPH cubic in $\mathbb{R}^{1,1}$ with exactly two different points with light–like tangents (marked by the grey circles).

There are no MPH cubics in $\mathbb{R}^{1,1}$ *except for the curves* $\mathbf{q}_1(t)$, $\mathbf{q}_2(t)$ *and straight lines.*

Proof: Let us suppose that $\mathbf{c}(t) = (x(t), y(t))^\top$ is an MPH cubic in $\mathbb{R}^{1,1}$. Then there exist two linear polynomials $u(t) = u_0(1-t)+u_1t$, $v(t) = v_0(1-t)+v_1t$ (cf. Kubota (1972)) such that

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t), \\ y'(t) &= 2u(t)v(t), \\ \sigma(t) &= u^2(t) - v^2(t), \end{aligned} \tag{27}$$

where σ is the parametric speed

$$\begin{aligned} &\sigma = (at - u_0 - v_0)(bt - u_0 + v_0), \text{ where} \\ &a = u_0 - u_1 + v_0 - v_1,\ b = u_0 - u_1 - v_0 + v_1. \end{aligned} \tag{28}$$

Depending on the number of light–like tangents of $\mathbf{c}(t)$ we will distinguish the following four cases.

Case 1: The curve $\mathbf{c}(t)$ has infinitely many light–like tangents. This case occurs when $\sigma(t) \equiv 0$, which implies that $\mathbf{c}(t)$ is a part of a light–like straight line.

Case 2: The curve $\mathbf{c}(t)$ has no light–like tangents. This means that $\sigma(t)$ has no roots, i.e. $a = 0$ and $b = 0$, see Eq. (28). One can easily verify that $\mathbf{c}(t)$ is a part of a straight line.

Case 3: The curve $\mathbf{c}(t)$ has exactly one light–like tangent. The limit case, when the two roots of σ degenerate into one, gives again straight lines only. Let us (without loss of generality) suppose that $a \neq 0$ and $b = 0$. Then $\mathbf{c}(t)$ has a light–like tangent at $t_0 = \frac{u_0+v_0}{a}$. We may assume that $t_0 = 0$ (otherwise we would reparameterize $\mathbf{c}(t)$) and therefore $u_0 + v_0 = 0$. A simple calculation reveals that

$$\mathbf{c}(t) = (\alpha t^3 + \beta t, \alpha t^3 - \beta t)^\top;\ \alpha = \frac{2}{3}(u_0 + v_1)^2,\ \beta = 2u_0^2. \tag{29}$$

By a reparameterization $t = t\sqrt{\frac{\beta}{3\alpha}}$ of $\mathbf{c}(t)$ given in (29) (the equality $\alpha = 0$ yields a straight line) and a scaling by factor $\sqrt{\frac{27\alpha}{\beta^3}}$ we obtain the curve $\mathbf{q}_1(t)$.

Case 4: The curve $\mathbf{c}(t)$ has exactly two different light–like tangents corresponding to $t_1 = \frac{u_0+v_0}{a}$ and $t_2 = \frac{v_0-u_0}{b}$, $a \neq 0$, $b \neq 0$ and $t_1 \neq t_2$. Let us consider the following transformation of the curve $\mathbf{c}(t) = (x(t), y(t))^\top$ consisting of a reparameterization $t = t + \delta$, a scaling by factor λ, a hyperbolic rotation with a hyperbolic angle φ and a translation given by the vector $(\varrho_1, \varrho_2)^\top$:

$$\mathbf{p}(t) = \lambda \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \mathbf{c}(t+\delta) + \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix}. \tag{30}$$

A straightforward but long computation gives that for the values

$$\varphi = \frac{1}{2} \ln \frac{b^2}{a^2},\ \delta = \frac{u_0^2 - u_0 u_1 + v_0 v_1 - v_0^2}{ab},\ \lambda = 3a^2 \sqrt{\frac{b^2}{a^2}} \tag{31}$$

and ϱ_1, ϱ_2 such that $\mathbf{p}(0) = (0,0)^\top$, the curve becomes

$$\mathbf{p}(t) = (a^2 b^2 t^3 + 3c^2 t, 3abct^2)^\top;\ c = u_1 v_0 - v_1 u_0. \tag{32}$$

A reparameterization $t = t\sqrt{\frac{c^2}{a^2b^2}}$ of $\mathbf{p}(t)$ given in (32) and a scaling by factor $\frac{1}{c^2}\sqrt{\frac{a^2b^2}{c^2}}$ gives the curve $\mathbf{q}_2(t)$. The equality $c = 0$ obviously leads to a straight line. □

6 Classification of spatial space–like MPH cubics

The following classification of spatial space–like MPH cubics is based on the notion of tangent indicatrix, i.e., the curve on the unit hyperboloid describing the variation of the unit tangent vector.

6.1 Orthogonal projections into planes perpendicular to the axis

Let $\mathbf{c}(t)$ be a space–like MPH cubic with curvature $\kappa \neq 0$ and torsion $\tau \neq 0$ and let $\mathbf{T}$ and $\mathbf{B}$ be the unit tangent and binormal vector of $\mathbf{c}(t)$. From the proof of Lancret's theorem (cf. Proposition 9) follows that the direction of the axis of $\mathbf{c}(t)$ (considered as a helix in $\mathbb{R}^{2,1}$) is given by the vector $\mathbf{v} = \mathbf{T} - \frac{\kappa}{\tau}\mathbf{B}$.

One can easily verify that $\langle \mathbf{T}, \mathbf{v} \rangle = 1$ and $||\mathbf{v}||^2 = 1 + \frac{\kappa^2}{\tau^2}||\mathbf{B}||^2$. Therefore, when $\mathbf{B}$ is space–like, the vector $\mathbf{v}$ is space–like as well. In the case when $\mathbf{B}$ is time–like, the causal character of $\mathbf{v}$ may be arbitrary, e.g. the axis of $\mathbf{c}(t)$ is light–like if and only if $||\mathbf{B}||^2 = -1$ and $\tau = \pm\kappa$.

In the Euclidean space one can use the following approach for constructing spatial PH cubics (cubic helices). It is obvious that an orthogonal projection of a PH cubic to a plane perpendicular to its axis is a planar PH cubic (since the length of the tangent vector of the projection is a constant multiple of the length of the tangent vector of the original curve), i.e. the Tschirnhausen cubic. Therefore all PH cubics can be obtained as helices "over" the Tschirnhausen cubic by choosing its slope (or equivalently the constant ratio of its curvature to torsion). Consequently, there exists only one spatial PH cubic up to orthogonal transforms and 1D scalings in $\mathbb{R}^3$, see Farouki and Sakkalis (1994). Unfortunately, in Minkowski space, this approach does not include the case of light–like axes.

Remark 16 Let $\mathbf{c}(t) = (x(t), y(t), r(t))^\top$ be a spatial space–like MPH cubic whose axis is space–like. We can suppose without loss of generality that its axis is the y axis. Then its hodograph satisfies $x'^2(t) + y'^2(t) - r'^2(t) = \sigma^2(t)$ for some polynomial $\sigma(t)$ and the orthogonal projection of $\mathbf{c}(t)$ to the xr plane is the curve $\mathbf{c}_0(t) = (x(t), 0, r(t))^\top$. As

$$|\sigma(t)| = \sqrt{x'^2(t) + y'^2(t) - r'^2(t)} = \lambda\sqrt{|x'^2(t) - r'^2(t)|} \tag{33}$$

for some constant $\lambda \neq 0$, there exists a polynomial $\sigma_0(t)$ such that $x'^2(t) - r'^2(t) = \pm\sigma_0^2(t)$. Therefore, the orthogonal projection of a spatial space–like MPH cubic in the direction of its axis is either a planar MPH cubic or a planar *time–like* MPH cubic satisfying $x'^2(t) - r'^2(t) = -\sigma_0^2(t)$. The notion of time–like MPH curve can be generalized to space, however, there is no application for it so far. In the planar case, one can think of time–like MPH curves as of space–like MPH curves, but with swapped space and time axis. Consequently, for the remainder of the paper we include planar time–like MPH curves into planar MPH curves, as no confusion is likely to arise.

In the case of a time–like axis (the r axis) of a spatial space–like MPH cubic the orthogonal projection in the direction of its axis is a planar PH cubic, i.e. the Tschirnhausen cubic.

6.2 Classification and normal forms

In particular, these results concerning the tangent indicatrices apply to space–like MPH cubics (with up to two points with light–like tangents). However, in this special case of MPH curves of degree 3, more can be achieved. As (space–like) MPH cubics are curves of a constant slope (see Proposition 9), their tangent indicatrix is a planar (conic) section of the unit hyperboloid.

Lemma 17 *Depending on its causal character, any plane π can be mapped using Lorentz transforms to one of the canonical positions shown in Table 1.*

Proof: Let us consider a plane π given by $ky + lr + m = 0$, $k \geq 0$, $l \leq 0$, $(k, l) \neq (0, 0)$, $m > 0$ (otherwise we rotate it about the time–axis and/or mirror it with respect to the xy plane). Depending on its causal character we transform π using hyperbolic rotations introduced in Table 1, cf. Figure 5. □

Theorem 18 *Any spatial space–like MPH cubic is similar to one of the curves listed in Table 2.*

Table 1
Canonical positions of a plane π, see Figure 5. The abbreviations sl., tl., and ll. stand for space–, time– and light–like, respectively.

π	Condition	Hyperbolic angle	Canonical position	Conic section (Euclid. classif.)
sl.	$k < -l$	$\frac{1}{2}\ln(-\frac{k+l}{k-l})$	π_s: $r = r_0$	'circle'
tl.	$k > -l, k^2 - l^2 \neq k^2$	$\frac{1}{2}\ln(\frac{k+l}{k-l})$	π_t: $y = y_0 \neq \pm 1$	'hyperbola'
	$k > -l, k^2 - l^2 = k^2$	$\frac{1}{2}\ln(\frac{k+l}{k-l})$	$\tilde{\pi}_t$: $y = \pm 1$	2 int. lines
ll.	$k = -l, m \neq 0$	$\ln \frac{m}{k}$	π_l: $y - r + 1 = 0$	'parabola'
	$k = -l, m = 0$	0	$\tilde{\pi}_l$: $y - r = 0$	2 par. lines

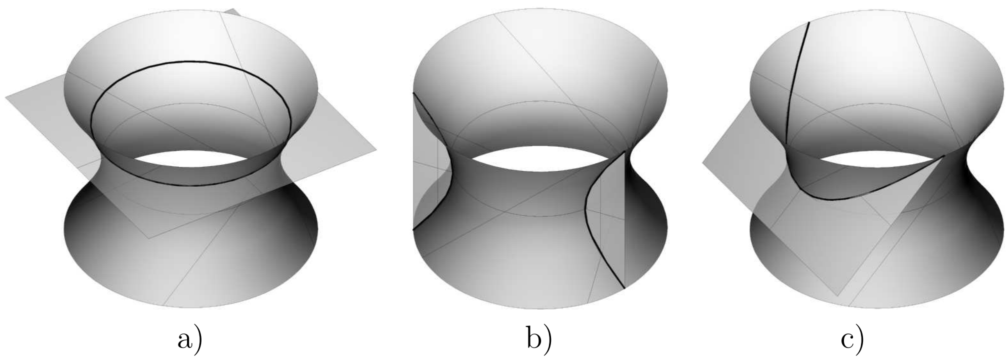

Fig. 5. The unit hyperboloid $\mathcal{H}$, canonical positions of a plane π and corresponding conic sections (Euclidean classification): a) space–like plane π_s and a 'circle' $\mathcal{K}_s$ b) time–like plane π_t and a 'hyperbola' $\mathcal{K}_t$ c) light–like plane π_l and a 'parabola' $\mathcal{K}_l$.

Proof: We distinguish two cases depending on whether the axis of the MPH cubic is light–like or not.

Case 1: Let $\mathbf{p}(t)$ be a spatial space–like MPH cubic whose axis is not light–like. Its orthogonal projection in the direction of its axis is a planar MPH cubic (cf. Remark 16). Now, we make use of the classification of planar MPH cubics described in Section 5.

Subcase 1.1: The axis is time–like. We have to analyze the MPH cubics "over" the Tschirnhausen cubic $\mathbf{T}(t) = (3t^2, t - 3t^3)^\top$.

Let $\mathbf{p}(t) = (3t^2, t - 3t^3, r(t))^\top$, where $r(t) = at^3 + bt^2 + ct$. Then we have $\mathbf{p}'(t) = (6t, 1 - 9t^2, 3at^2 + 2bt + c)^\top$ and

$$||\mathbf{p}'(t)||^2 = (\alpha t^2 + \beta t + \gamma)^2 \tag{34}$$

Table 2
Normal forms of spatial space–like MPH cubics.

Axis	Normal form	1D scaling factor	κ, τ
Space–like	$(t^3+3t, \lambda(t^3-3t), 3t^2)^\top$	$\lambda \neq 0$	$\tau = -\lambda\kappa$
	$(3t^2, \lambda(t^3-3t), t^3+3t)^\top$	$\|\lambda\| > 1$	$\tau = -\lambda\kappa$
Time–like	$(3t^2, t-3t^3, \lambda(3t^3+t))^\top$	$\|\lambda\| < 1$, $\lambda \neq 0$	$\tau = \lambda\kappa$
Light–like	$\frac{1}{6}(3t^2, t^3-6t, t^3)^\top$		$\kappa = -\tau = 1$
	$(3t^2, t^3-6t, 6t)^\top$		$\tau = -\kappa$

must hold for some constants α, β and γ. Comparing the coefficients in (34) gives the following system of equations:

$$\begin{aligned} &\alpha^2+9a^2-81=0, \;\; \alpha\beta+6ab=0, \;\; \beta^2+2\alpha\gamma+4b^2+6ac-18=0, \\ &\beta\gamma+2bc=0, \;\; \gamma^2+c^2-1=0. \end{aligned} \tag{35}$$

All solutions of the form (a,b,c) of (35) are found to be $(-3\lambda, 3\sqrt{1-\lambda^2}, \lambda)$ and $(3\lambda, 0, \lambda)$, $|\lambda| < 1$. However, as the first family of solutions gives only planar cubics, the only spatial space–like MPH cubics with time–like axis are given by

$$\mathbf{p}(t) = (3t^2, t-3t^3, \lambda(3t^3+t))^\top, \;\; |\lambda| < 1, \;\; \lambda \neq 0. \tag{36}$$

Note that there exists only one spatial space–like MPH cubic with time–like axis up to Minkowski similarities and 1D scalings given by the factor λ. For $\lambda = 0$ one obviously obtains a planar MPH cubic and the limit cases $\lambda = \pm 1$ give the W–null–cubic.

Subcase 1.2 The axis is space–like. All spatial space–like MPH cubics with space–like axes can be found analogously with the help of the planar MPH cubics presented in Proposition 15. For the sake of brevity we omit the details.

Case 2: On the other hand, let $\mathbf{q}(t)$ be a spatial space–like MPH cubic whose axis is light–like. Since the previous construction (based on Remark 16) cannot be used in this case, we turn our attention to the so–called tangent indicatrix

$$\mathbf{r}(t) = \frac{\mathbf{q}'(t)}{||\mathbf{q}'(t)||}. \tag{37}$$

Since the axis is light–like, the tangent indicatrix is contained in the intersection of a certain light–like plane with the unit hyperboloid $\mathcal{H}$. Without loss of generality we consider the light–like plane π_l: $y - r + 1 = 0$, cf. Table 1, Fig. 5c. (The other case of light–like planes can be omitted, since the tangent indicatrix would be contained in one of two parallel lines. This is only possible for planar curves).

The tangent indicatrix $\mathcal{K}_l = \pi_l \cap \mathcal{H}$ has the parametric representation $\mathbf{r}(t) = (t, \frac{t^2}{2} - 1, t^2)^\top$. All other rational biquadratic parameterizations are obtained by the bilinear reparameterizations $t = (a\tau + b)/(c\tau + d)$, $ad - bc \neq 0$ (cf. Farin (1987)) of $\mathbf{r}(t)$,

$$\tilde{\mathbf{r}}(\tau) = \left(\frac{a\tau + b}{c\tau + d}, \frac{(a\tau + b)^2 - 2(c\tau + d)^2}{2(c\tau + d)^2}, \frac{(a\tau + b)^2}{2(c\tau + d)^2} \right)^\top . \tag{38}$$

Each reparameterization leads to an MPH cubic, which is obtained by integrating the numerator, after introducing a common denominator,

$$\tilde{\mathbf{q}}(\tau) = \begin{pmatrix} \tau(2ca\tau^2 + 3\tau da + 3\tau cb + 6db) \\ \tau(a^2\tau^2 + 3a\tau b + 3b^2 - 2c^2\tau^2 - 6c\tau d - 6d^2) \\ \frac{1}{a}(a\tau + b)^3 \end{pmatrix}^\top . \tag{39}$$

Subcase 2.1: $c \neq 0$. A straightforward computation shows that we have

$$\frac{4c^4}{(bc - ad)^3} L\tilde{\mathbf{q}} \left(\frac{ad - bc}{2c^2} t - \frac{d}{c} \right) + (\tau_1, \tau_2, \tau_3)^\top = (3t^2, t^3 - 6t, 6t)^\top, \tag{40}$$

where L is a Lorentz transform

$$L = \frac{1}{2c^2} \begin{pmatrix} 2c^2 & 2ca & -2ca \\ -2ca & 2c^2 - a^2 & a^2 \\ 2ca & a^2 & -2c^2 - a^2 \end{pmatrix} \tag{41}$$

and $(\tau_1, \tau_2, \tau_3)^\top$ is a translation vector computed from $\mathbf{q}(0) = (0, 0, 0)^\top$.

Subcase 2.2: $c = 0$. In this case one obtains that

$$\frac{a}{d^3} \mathbf{q} \left(\frac{d}{a} t - \frac{b}{a} \right) + (\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3)^\top = (3t^2, t^3 - 6t, t^3)^\top . \tag{42}$$

This completes the proof. □

The results are summarized in Table 2 including the ratios of curvatures to torsions of the spatial space–like MPH cubics. In the case of a space–like or time–like axis, two families or one family of space–like MPH cubics exist, respectively. In addition, there are two space–like MPH cubics with light–like axis.

7 Spatial light–like MPH cubics: The W–null–cubic

We have already shown that the only spatial light–like MPH cubic is the W–null–cubic $\mathbf{w}(t) = (3t^2, t - 3t^3, t + 3t^3)^\top$, compare with (1) and (15). According to Wunderlich (1973), it is the normal form of the only cubic helix (for constant slope $\alpha = \pi/4$) in Euclidean space. Moreover, as shown by Wagner and Ravani (1997), it is also the only so–called cubic RF curve, i.e. a polynomial cubic with rational Frenet–Serret motion (of degree 5) in Euclidean space.

The orthogonal projections of the W–null–cubic to into the xy, yr and xr planes are again PH or (time–like) MPH curves (see Fig. 6). The projections are similar

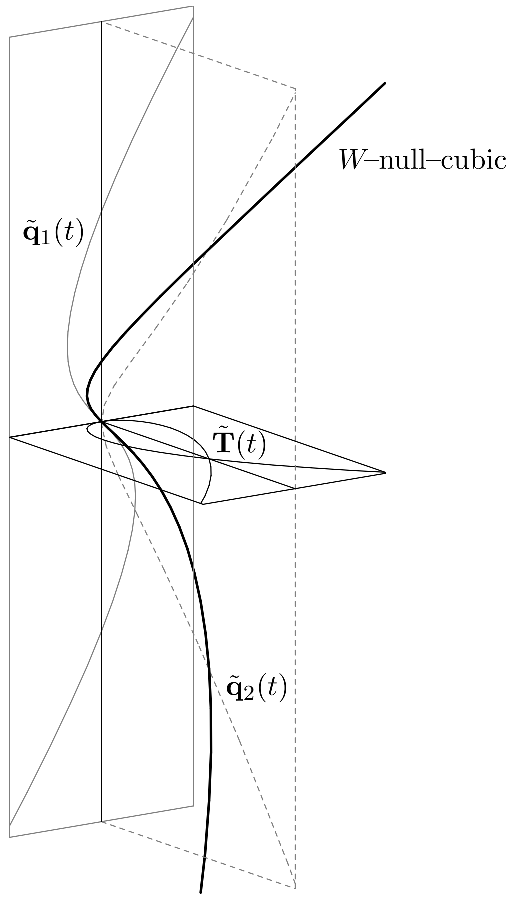

Fig. 6. The W–null–cubic and its projections into the coordinate planes.

to the Tschirnhausen cubic $\mathbf{T}(t) \in \mathbb{R}^2$ (see Section 5, Fig. 2), to the curve with one light–like tangent $\mathbf{q}_1(t) \in \mathbb{R}^{1,1}$ (cf. Proposition 15, Fig. 3) and to the curve with two light–like tangents $\mathbf{q}_2(t) \in \mathbb{R}^{1,1}$ (cf. Proposition 15, Fig. 4), respectively.

8 Conclusion

In this paper we investigated a relation between MPH cubics and cubic helices in Minkowski space. Among other results we proved that any polynomial space–like or light–like helix in $\mathbb{R}^{2,1}$ is an MPH curve. The converse result holds for cubic MPH curves, i.e. spatial MPH cubics are helices in $\mathbb{R}^{2,1}$. Based on these results and properties of tangent indicatrices of MPH curves we presented a complete classification of planar and spatial MPH cubics.

Acknowledgement The authors thank the Austrian Science fund (FWF) for supporting this research through project P17387-N12.

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