# Approximating Offsets of Surfaces by using the Support Function Representation

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**Summary.** The support function (SF) representation of surfaces is useful for analyzing curvatures and for representing offset surfaces. After reviewing basic properties of the SF representation, we discuss several techniques for approximating the SF of a given surface.

## 1 Introduction

Robust and efficient methods for dealing with offset curves and surfaces are one of the major challenges in Computer Aided Design. Offset to (piecewise) rational curves and surfaces (i.e., NURBS) are not rational and need to be approximated. Also, singularities and self–intersections can easily be generated and have to be dealt with [Mae].

Certain subsets of the set of rational curves and surfaces are closed under offsetting, or even under the (more general) convolution operator [PP]. In particular, such subsets can be obtained by using the support function (SF) representation, where the support functions vary in the space of polynomials [SGJ1]. The SF representation is one of the classical tools in the field of convex geometry, see e.g. [Gro]. Its application to problems in Computer Aided Design can be traced back to a classical paper of Sabin [Sab]. It does not only provides computational advantages for dealing with offsets, but also leads to particularly simple expressions for quantities and mappings governing the differential geometry of surfaces.

## 2 Support function representation of surfaces

For any smooth surface $\Sigma$ in three–dimensional space, the so–called *Gauss map* $\gamma : \Sigma \to \mathbb{S}^2$ assigns to each point $\mathbf{x} \in \Sigma$ the associated unit normal $\mathbf{n}(\mathbf{x})$, which is identified with a point on the unit sphere, cf. Fig. 1. It can be used to analyze the curvature of the surface. In particular, the Weingarten map equals

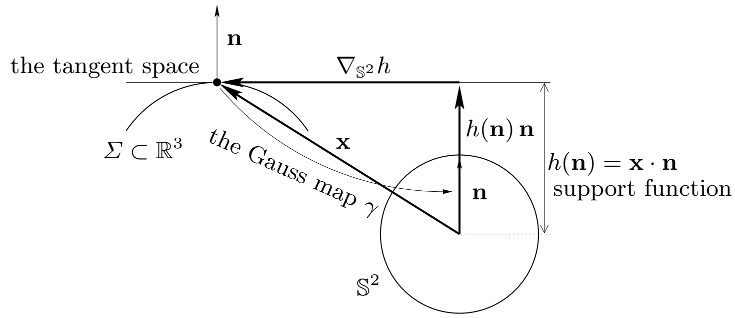


**Fig. 1.** The Gauss map and the SF of a surface.

$-\mathrm{d}\gamma$ and the principal curvatures and principal directions are its eigenvalues and eigenvectors, respectively. The Gaussian curvature is the product of the principal curvatures, i.e., the determinant of the Weingarten map. So if the Gaussian curvature does not vanish, then the Weingarten map is invertible and Gauss map is locally invertible.

Consequently, any surface with non–vanishing Gaussian curvature can locally be described by its inverse Gauss map. Since the Gauss map is geometrically significant, many geometric constructions simplify if its inverse is explicitly known. The function

$$h_0 : \Sigma \to \mathbb{R} : \mathbf{x} \mapsto \mathbf{x}\cdot\mathbf{n}(\mathbf{x}) \tag{1}$$

associates with each point the distance of its tangent plane to the origin. The *support function* (SF) $h : \mathbb{S}^2 \to \mathbb{R}$ is then obtained by composing this function with the inverse Gauss map, $h = \gamma^{-1} \circ h_0$. Under certain technical assumptions, the surface can be reconstructed from its SF (cf. [Gra, SGJ1]):

**Theorem 1.** *Let $U$ be an open subset of the unit sphere and $h \in C^k(U, \mathbb{R})$, where $k > 2$. Define $\mathbf{x}_h \in C^{k-1}(U, \mathbb{R}^3)$ by*

$$\mathbf{x}_h(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \nabla_{\mathbb{S}^2} h\big|_{\mathbf{n}}\,, \tag{2}$$

*where $\nabla_{\mathbb{S}^2}$ denotes the intrinsic gradient. If $\det(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})$ does not vanish in $U$, where $\mathrm{Hess}_{\mathbb{S}^2}(h)$ denotes the intrinsic Hessian of $h$, then*

1. *The image $\mathbf{x}_h(U)$ is a $C^k$-surface and its SF is $h$.*
2. *The Weingarten map of the surface is $-(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})^{-1}$.*
3. *If $\lambda$ is an eigenvalue of $\mathrm{Hess}_{\mathbb{S}^2}(h)$ and $\mathbf{e}$ the associated eigenvector, then $-1/(h+\lambda)$ is a principal curvature and $\mathbf{e}$ is a principal curvature direction.*
4. *The Gaussian and the mean curvatures are*

$$K = \frac{1}{\det(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})}, \quad M = \frac{-\operatorname{tr}(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})}{2\det(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})} \tag{3}$$

5. *Point-wise the absolute value of $h$ and the norms of its gradient and $\mathbf{x}_h$ are related by*
$$\|\mathbf{x}_h(\mathbf{n})\|^2 = h(\mathbf{n})^2 + \|\nabla_{S^2} h(\mathbf{n})\|^2, \tag{4}$$
6. *The $L^2$ norms of $h$ and $\mathbf{x}_h$ are related by $\|\mathbf{x}_h\|_2^2 = \|h\|_2^2 + \|\nabla_{\mathbb{S}^2} h\|_2^2$.*
7. *The maximum norms satisfy $\|\mathbf{x}_h\|_\infty^2 \leq \|h\|_\infty^2 + \|\nabla_{\mathbb{S}^2} h\|_\infty^2$. In particular, if $U = \mathbb{S}^2$ and the surface $\mathbf{x}_h$ is regular everywhere, then this inequality becomes an equation.*

The SF of a surface behaves nicely under geometrical transformations. Translation and offsetting correspond to adding linear and constant functions, respectively, while rotations have to be composed with $h$. Consequently, the maximum allowed offsetting distance which does not introduce self–intersections or singularities can be computed by analyzing the eigenvalues of the Hessian.

Note that the mapping $h \to \mathbf{x}_h$ is linear; it introduces an isomorphism between the linear spaces $C^k(U, \mathbb{R})$ and its images, where the addition in the image spaces is given by the so–called convolution (in the sense of [SPJ]) of surfaces, see [SGJ2].

The linearity implies in particular that the norm estimates above are invariant under offsetting.

If $k = 1$, then the Hessian cannot be used to analyze the regularity. However, if $h$ is globally $C^1$ and piecewise $C^2$ and the sign of $\det(\mathrm{Hess}_{\mathbb{S}^2}(h) + h\,\mathrm{id})$ is the same on each patch, then the surface is of class $C^1$, see [Gra, SGJ1].

## 3 Approximation of surfaces

According to results 6 and 7 of the theorem, we can translate questions concerning approximation of surfaces with non–vanishing Gaussian curvature to questions concerning the approximation of scalar fields on $\mathbb{S}^2$, cf. [ANS].

*Approximation by harmonic expansions.* If we consider a surface whose support is either defined or can smoothly be extended to $\mathbb{S}^2$, then it is possible to apply the tools from harmonic analysis. Note that the harmonic expansion leads to rational surfaces with rational offsets. Indeed, by composing the harmonic expansion with a rational parameterization of the sphere, Eq. (2) gives a rational parametric representation, which complies with the CAD standard.

This applies immediately to closed convex surfaces, which are studied in convex geometry (see Example 22 of [SGJ1]). Here we present a non–convex one. We consider a one–sheeted hyperboloid of revolution with the support function $h_0 = \sqrt{x^2 + y^2 - z^2}$. In order to approximate this surface and its offsets, we restrict $h_0$ to the sphere zone $|z| \leq \frac{1}{2}\sqrt{2} - \epsilon$, where $\epsilon$ is a small constant, and extend the restriction to a function $h^* \in C^3(\mathbb{S}^2, \mathbb{R})$. The results are shown in Fig. 2.

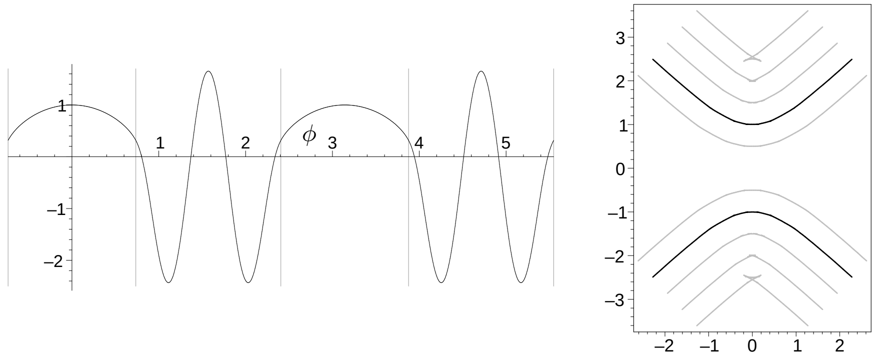


**Fig. 2.** Support function of a non–convex surface of revolution (the concave region between the vertical grey bars) and its $C^3$ smooth extension (left). Approximation of the surface and of its offsets (right). In both cases, only the intersections with the plane $y = 0$ are shown, and the support function is parameterized by the angle.

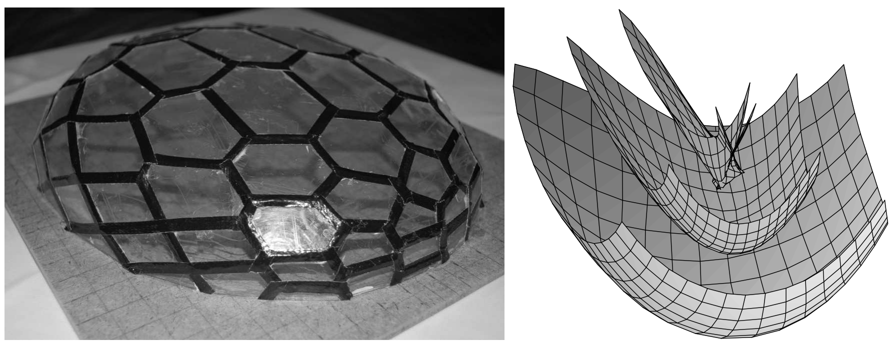

**Fig. 3.** Approximations constructed via the SF.

*Approximation by piecewise linear functions.* Another very interesting way to approximate the SF $h$ is by using a piecewise linear function $\overline{h}$ defined over a triangulation of (a part of) the unit sphere. Each vertex $\mathbf{n}_i$ defines the plane $\mathbf{x}\cdot\mathbf{n}_i = h_i = h(\mathbf{n_i})$ in $\mathbb{R}^3$. Each triangle defines a point $\mathbf{v} \in \mathbb{R}^3$ where the linear function $\mathbf{v} \cdot \mathbf{n}$ interpolates the values of the SF in the corners of the triangle. Clearly $\mathbf{v}$ is the point of intersection between the three planes defined by the corners of the triangle.

The triangles around a vertex define a polygon in the plane defined by the vertex. We obtain a graph embedded in $\mathbb{R}^3$ with planar faces which is the dual to the triangulation. Fig. 3 left shows a photograph of a physical model of a surface with planar faces approximating half of an ellipsoid. The technique can be applied to non–convex surfaces too, see [SGJ1].

Note that the planar faces may have self–intersections (‘swallowtails’). In order to avoid these problems, the spherical triangulation may have to be modified by ‘edge flipping’.

*Least–squares fitting.* In many cases the SF is not (explicitly) available and only a surface patch or point cloud may be given. For these cases we propose following approximation scheme, which is to be applied to a given surface represented by sample points $\mathbf{X}_i$, possibly with associated normals $\mathbf{n}_i$.

1. Sample points $\mathbf{X}_i$ and associated unit normals $\mathbf{n}_i$ from the patch. If the points $\mathbf{X}_i$ are the input, then estimates the normal $\mathbf{n}_i$ (e.g., based on local planes of regression).
2. Consider a suitable[3] finite-dimensional space $\mathcal{H}$ of support functions with basis $h_j$.
3. Find the SF $h = \sum_j \alpha_j h_j$ such that the associated surface $\mathbf{x}_h$ approximates the data in the least–squares sense, by minimizing the objective function

$$\sum_{i=1}^{N}\left(\left(\mathbf{X}_i \cdot \mathbf{n}_i - \sum_j \alpha_j h_j(\mathbf{n}_i)\right)^2 + \left\|\mathbf{X}_i - (\mathbf{X}_i \cdot \mathbf{n}_i)\mathbf{n}_i - \sum_j \alpha_j \nabla_{\mathbb{S}^2} h_j \big|_{\mathbf{n}_i}\right\|^2\right).$$

As an example, we approximated the support function of a biquadratic patch by a support function of degree 9, see Fig. 3, right. In this case, 256 sample points were used in order to define the objective function. In the same picture two offsets are also depicted and it is an important fact that they are approximated by exactly the same precision as the surface itself.

## References

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[3] In order to be invariant with respect to translations and offsetting this space should contain all polynomials of degree 1.