Multiresolution Analysis for Implicitly Defined Algebraic Spline Curves with Weighted Wavelets

Mario Kapl and Bert Jüttler

Abstract. We describe a method to construct a hierarchical representation of a given implicitly defined algebraic spline curve with the help of weighted spline wavelets. These wavelets are adapted to the region of interest, in our case to the region along the curve, by means of a weighted inner product. The application of two different types of weighted spline wavelets is considered and compared with standard spline wavelets.

§1. Introduction

Spline wavelets are a powerful mathematical tool for hierarchically decomposing functions which combines the properties of splines and wavelets. Typical applications of spline wavelets include hierarchical visualizations of geometric objects, data compression and numerical simulation. For these applications, different types of spline wavelets have been constructed. Examples are spline wavelets on a bounded interval e.g. [2, 3, 5, 6, 13], compactly supported spline wavelets e.g. [4], wavelets with minimal support e.g. [9] and non-uniform spline wavelets e.g. [10].

We consider the space V^j of spline functions of degree d with period 1 and knots $\mathbb{Z}/(d+1)2^j$. Let $\phi_0^{j,d}, \dots, \phi_{\dim V^j - 1}^{j,d}$ be the B-spline basis of V^j. We call the basic functions of a spline function space W^j such that $V^{j+1} = V^j \oplus W^j$ *spline wavelets*, and denote these functions by $\psi_0^{j,d}, \dots, \psi_{\dim W^j - 1}^{j,d}$.

The spline wavelet construction is called *orthogonal* if the basic functions $\phi_i^{j,d}$ form an orthonormal basis of V^j, the wavelets $\psi_k^{j,d}$ form an orthonormal basis of W^j and the basic functions $\phi_i^{j,d}$ are orthogonal to the wavelets $\psi_k^{j,d}$. If the last condition is satisfied, we denote the spline

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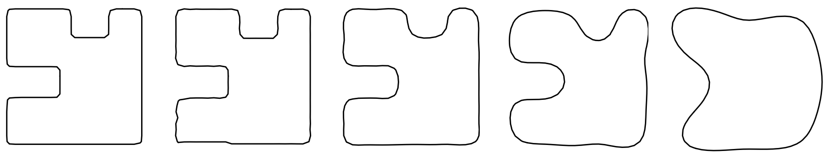


Fig. 1. Hierarchical representation of an implicitly defined algebraic spline curve

wavelets as *semiorthogonal*. Otherwise we call them *biorthogonal spline wavelets*.

A wavelet construction can be described with the help of four matrices, namely with the *analysis matrices* A^j, B^j and the *synthesis matrices* P^j, Q^j. Furthermore the analysis and synthesis matrices can be used to compute the coefficients and the detail information for the different levels (cf. [15]).

In the present paper we consider the following problem. Given an implicitly defined algebraic spline curve, we are interested in its hierarchical representation. An *implicitly defined algebraic spline curve* is the zero-contour $f^{(m,n)}(x,y) = 0$ of a tensor-product spline function of bi-degree (u,v)

$$f^{(m,n)}(x,y) = \sum_{k=0}^{\dim V^m - 1} \sum_{l=0}^{\dim V^n - 1} c_{k,l}^{(m,n)} \phi_k^{m,u}(x) \phi_l^{n,v}(y),$$

with $m \geq 1, n \geq 1, u \geq 1$ and $v \geq 1$, with coefficients $c_{k,l}^{(m,n)} \in \mathbb{R}$ which are called *control points* of the the tensor-product spline function $f^{(m,n)}$. The upper indices (m,n) refer to the level of detail of representation.

In the following sections we show how to adapt spline wavelets to this application. Section 2 describes the concept of two different types of weighted spline wavelets. In Section 3 we give a method to construct a hierarchical representation of an implicitly defined algebraic spline curve with the help of these weighted wavelets. Section 4 compares the different weighted spline wavelets with standard spline wavelets on a concrete example. Finally we conclude this paper in Section 5.

§2. Weighted Spline Wavelets

We are interested in a spline wavelet construction that preserves the zero-contour of a tensor-product spline function as well as possible. The construction of these wavelets consists of two steps. First we construct wavelets for univariate spline functions. These wavelets have an increased "approximation power" in a certain region of the domain $[0, 1]$, which we call the region of interest. In the second step we apply these one-dimensional wavelets to the tensor-product spline representation of our function. In this section we explain the first construction step. The second step will be considered in the next section.

Let $D^j \subset [0, 1]$ be the region of interest, which is chosen as the union of intervals with the knots of the spline functions as endpoints. Let $w^j : [0, 1] \to \mathbb{R}$ be such that

$$w^j(x) := \begin{cases} 1, & \text{for } x \in [0,1] \setminus D^j, \\ u, & \text{for } x \in D^j, \end{cases}$$

where $u \in \mathbb{R}$ and $u > 1$. For a function $w : [0, 1] \to \mathbb{R}$ let $\langle \cdot | \cdot \rangle_w$ be the weighted inner product

$$\langle f|g\rangle_w := \int_0^1 w(x) \cdot f(x) \cdot g(x) \mathrm{d}x.$$

We construct spline wavelets that are adapted to the region of interest by means of a weighted inner product $\langle \cdot | \cdot \rangle_w$. We call these constructed spline wavelets *weighted spline wavelets*.

2.1. Types of weighted spline wavelets

We distinguish between two different types of weighted spline wavelets:

Weighted semiorthogonal spline wavelets (cf. [7]): These are spline wavelets $\psi_l^{j,d}$ that are semiorthogonal with respect to the weighted inner product $\langle \cdot | \cdot \rangle_{w^j}$ i.e. $\langle \phi_k^{j,d} | \psi_l^{j,d} \rangle_{w^j} = 0$ for all $k \in \{0, \ldots, \dim V^j - 1\}$, $l \in \{0, \ldots, \dim W^j - 1\}$ and $j \in \mathbb{N}_0$. The construction of these wavelets is based on selecting a synthesis matrix Q^j such that

$$(P^j)^T (\langle \phi_k^{j,d} | \phi_l^{j,d} \rangle_{w^j})_{k,l} Q^j = 0.$$

The choice of Q^j determines the wavelets $\psi_0^{j,d}, \ldots, \psi_{\dim V^j - 1}^{j,d}$. A disadvantage of (weighted) semiorthogonal spline wavelets is that there is no construction known such that all four matrices P^j, Q^j, A^j and B^j have band structure (cf. [15]).

Weighted biorthogonal spline wavelets (cf. [8]): These are wavelets that are constructed with the help of lifting, a general method for modifying an existing biorthogonal wavelet construction (cf. [12, 14, 15]). The construction of the weighted biorthogonal spline wavelets consists of two steps. As first step we construct lazy spline wavelets (cf. [8, 14]) with a small support. These are biorthogonal spline wavelets with banded analysis and synthesis matrices. In the second step we modify these lazy

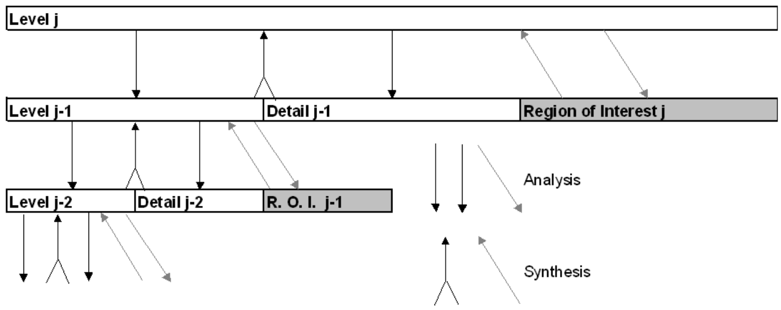


Fig. 2. The wavelet transform (black) and the weighted wavelet transform (black and grey).

wavelets by increasing the L^2-orthogonality with respect to a weighted inner product $\langle\cdot|\cdot\rangle_{w^j}$. That means we use the lazy spline wavelets to construct biorthogonal spline wavelets $\psi_k^{j,d}$ such that

$$\sum_k \sum_l \langle \phi_k^{j,d} | \psi_l^{j,d} \rangle_{w^j}$$

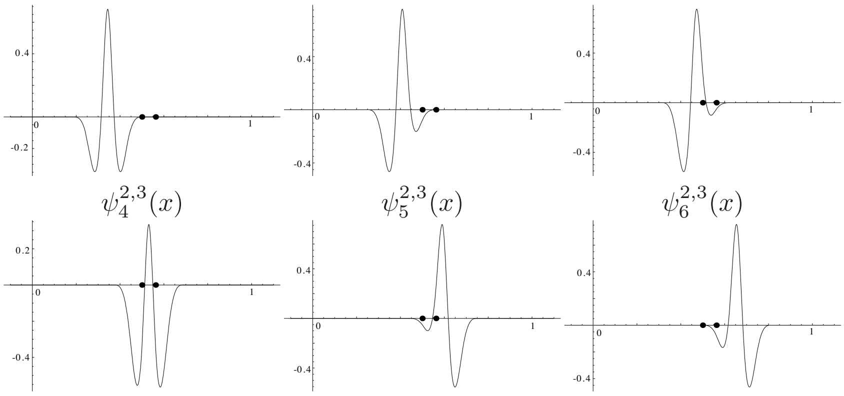
is minimized. An advantage of these wavelets is that the analysis and synthesis matrices are still banded but only their bandwidths have increased.

2.2. Properties of weighted spline wavelets

The wavelet transform for weighted wavelets differs from the standard wavelet transform in one point, see Figure 2. In addition to the coefficients and detail information in level $j-1$, we need for the reconstruction of the coefficients in level j the information about the region of interest in level j. The implementation of these wavelets can be done in the usual way for wavelets. Additionally a pointer to the region of interest is needed.

By choosing D^j as union of intervals with the knots of the spline function as endpoints, we assure that only a small number of different weighted wavelets has to be considered. All other weighted wavelets can be constructed from them with the help of translation and scaling. A concrete example for weighted wavelets is given in Figure 3. The computational time for analysis and synthesis is comparable with the standard spline wavelet case because the analysis and synthesis matrices for the different weighted spline wavelets can be precomputed (use of "templates").

More precisely, we precompute the analysis and synthesis matrices for the standard spline wavelets (weight $u = 1$). In order to obtain the analysis and synthesis matrices for the different weighted wavelets, we have



$\psi_4^{2,3}(x)$ $\psi_5^{2,3}(x)$ $\psi_6^{2,3}(x)$

$\psi_7^{2,3}(x)$ $\psi_8^{2,3}(x)$ $\psi_9^{2,3}(x)$

Fig. 3. Weighted biorthogonal wavelets $\psi_i^{j-1,3}$ for 1-periodic uniform B-splines of degree 3 for $j = 3$ with $D^j = [\frac{4\cdot 2^{j-2}}{4\cdot 2^{j-1}}, \frac{4\cdot 2^{j-2}+1}{4\cdot 2^{j-1}}]$ and $u = 10$. Only $\psi_5^{2,3}(x), \psi_6^{2,3}(x), \psi_7^{2,3}(x), \psi_8^{2,3}(x)$ and $\psi_9^{2,3}(x)$ differ from the "standard" lifted wavelet $\psi_4^{2,3}(x)$ (u=1). The two dots mark the boundaries of the interval D^j.

only to replace some columns or rows of these precomputed matrices by corresponding precomputed columns or rows.

Experimental results have shown that a good choice for the weight u is between 5 and 10. If the value u is higher, then the analysis process may produce additional roots. On the other hand if the weight u is too low, then the effect of the weighted wavelets is small.

Different kinds of stability have been considered e.g. in [7, 8]. The weighted spline wavelets are *uniformly stable*, but numerical experiments indicate that *Riesz stability* is not to be expected.

§3. Hierarchical Representation of Implicitly Defined Algebraic Spline Curves

In order to obtain a hierarchical representation of an implicitly defined algebraic spline curve $f^{(m,n)}$ we apply the one-dimensional wavelet transform to the rows and columns of the coefficient matrix $(c_{k,l}^{(m,n)})_{k,l}$ of the tensor-product spline representation of $f^{(m,n)}$. There are two methods for decomposition quite well known (cf.[1] and [15]):

Standard decomposition: We apply the one-dimensional wavelet transform to each row, then we apply the one-dimensional wavelet transform to each column. This leads for a function $f^{(m,n)}$ to the following hierarchical representation: $f^{(m-1,n)}$, $f^{(m-2,n)}$, …, $f^{(0,n)}$, $f^{(0,n-1)}$, …, $f^{(0,0)}$.

Nonstandard decomposition: We apply one step of the one-dimensional wavelet transform to each row, then one step to each column and so on. We obtain for a function $f^{(m,n)}$ the following hierarchical representation: $f^{(m-1,n)}$, $f^{(m-1,n-1)}$, $f^{(m-2,n-1)}$, ..., $f^{(0,0)}$.

In this work we have used the nonstandard decomposition method because we are interested in a "geometric hierarchy". That is, we consider mainly the representations $f^{(i,i)}$, where a similiar amount of simplification has taken place along both coordinate axes. The standard decomposition would simplify along one axis as much as possible, before simplifying along the other axis.

Now in the case of weighted wavelets one step of the construction of the hierarchical representation works as follows. For each row or column of our tensor-product spline representation, we compute the region of interest D^j. This is done in the following way. We consider the coefficients of each row or column as a control polygon of a univariate function g and compute the roots of g numerically. For this we evaluate the values $g(\frac{i-1}{n})$ for some $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ and detecting sign changes of these values. In our case we have chosen $n = 10000$. For every root η we compute a corresponding interval $I(\eta)$ which contains the root. If $\eta \in [\frac{2i-1}{(d+1)2^{j+1}}, \frac{2i+1}{(d+1)2^{j+1}}[$, then we choose for $I(\eta)$ the interval $[\frac{i-1}{(d+1)2^j}, \frac{i+1}{(d+1)2^j}[$. We have used sampling because it is simple. For a more sophisticated computation of the roots we could use for example the method in [11]. Finally we take for the region of interest D^j the union of the computed intervals $I(\eta)$. Afterwards we apply the corresponding weighted spline wavelets to each of the rows or columns of the coefficient matrix.

We want to consider now more precisely what the representation of a decomposed function $f^{(m,n)}$ looks like. By applying one analysis step, the function $f^{(m,n)}$ is decomposed as

$$f^{(m-1,n)}(x,y) + \sum_{k=0}^{\dim W^{m-1}-1} \sum_{l=0}^{\dim V^n -1} d_{k,l}^{(m-1,n)} \psi_{k,l}^{m-1,u}(x) \phi_l^{n,v}(y),$$

where $d_{k,l}^{(m-1,n)}$ are the wavelet coefficients and $\psi_{k,l}^{m-1,u}$ are the weighted wavelets $\psi_k^{m-1,u}$, depending on l. That means for each row or column we can have different weighted wavelets and different wavelet spaces, respectively. Because we can use different wavelets for each row or column, we can adapt the two-dimensional wavelet transform to the shape of the curve.

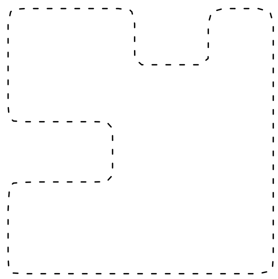


Fig. 4. Original curve at level 4.

§4. Comparison

In this section we compare the different weighted spline wavelets with standard spline wavelets on a concrete example.

Example 1. We consider the implicitly defined algebraic spline curve $f^{(4,4)}$ of bi-degree $(3,3)$ given in Figure 4 at level 4 (dashed curve). We compare now the preservation of this curve by applying different spline wavelet constructions. In Figure 5 we can see the resulting curves $f^{(3,3)}$ at level 3 and $f^{(2,2)}$ at level 2 by using weighted biorthogonal spline wavelets with a weight $u = 5$ and appropriate weighted regions (black curve) and by using standard lifted spline wavelets (grey curve), which can be understood as weighted biorthogonal wavelets with a weight $u = 1$. Remember, we get the resulting curves $f^{(3,3)}$ from $f^{(4,4)}$ by applying one analysis step to the rows of the coefficient matrix and then one analysis step to the columns of the coefficient matrix. The curves $f^{(2,2)}$ are obtained from $f^{(3,3)}$ by applying again one analysis step to the rows and one analysis step to the columns of the coefficient matrix.

The comparison of weighted semiorthogonal spline wavelets with a weight $u = 5$ and appropriate weighted regions (black curve) with standard semiorthogonal spline wavelets (grey curve) can be seen in Figure 6. The two different weighted wavelet constructions are compared in Figure 7.

§5. Conclusion

We have described a method to construct a hierarchical representation of a given implicitly defined algebraic spline curve with the help of weighted wavelets. For this we have explained the concept of two different types of weighted spline wavelets (semiorthogonal and biorthogonal). As we can see in Example 1 (Figures 5-7), the different weighted spline wavelets

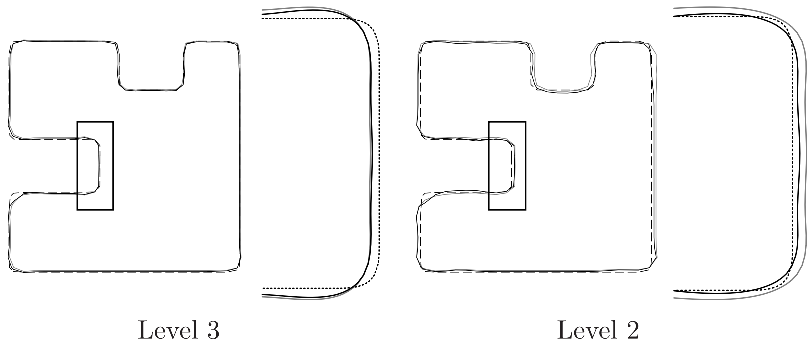


Fig. 5. Comparison of weighted biorthogonal wavelets (black) and standard lifted biorthogonal wavelets (grey) with the original curve (dashed).

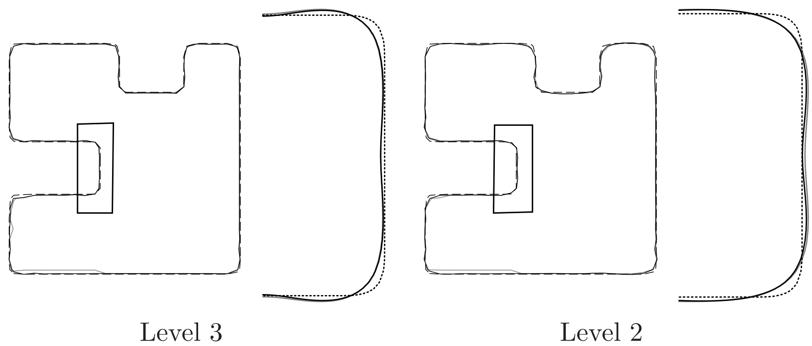


Fig. 6. Comparison of weighted semiorthogonal wavelets (black) and standard semiorthogonal wavelets (grey) with the original curve (dashed).

preserve a curve better than the corresponding standard spline wavelets. For standard spline wavelets, similiar results could be obtained by including smaller wavelet coefficients in the areas near the zero-contour. Using weighted spline wavelets, one can achieve a better representation without having to use smaller wavelet coefficients. Consequently, if one wishes to convert the result to a tensor-product representation with unform knots, a smaller number of knots is needed to achieve the same accuracy. Comparing the two different weighted wavelet constructions, we see that the weighted semiorthogonal wavelets are better than the weighted biorthogonal wavelets, but with the disadvantage of non-banded analysis matrices.

Clearly, the idea of our construction of a hierarchical representation of an implicitly defined algebraic spline curve with weighted wavelets can also be applied to implicitly defined surfaces or images with sharp features. This could be the subject of future work.

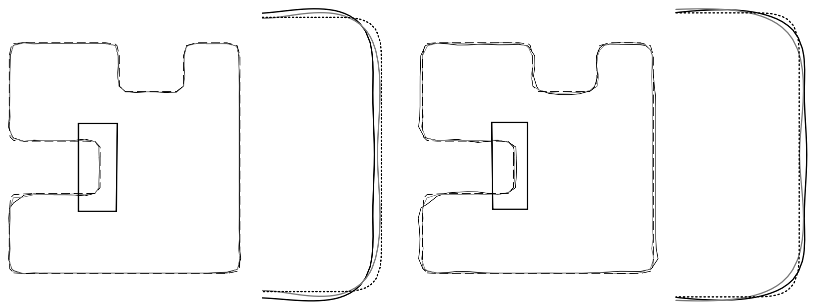


Fig. 7. Comparison of weighted biorthogonal wavelets (black) and weighted semiorthogonal wavelets (grey) with the original curve (dashed).

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