# Computing Convolutions and Minkowski Sums via Support Functions

Zbyněk Šír, Jens Gravesen and Bert Jüttler

**Abstract.** The convolution of two simple closed oriented curves or surfaces, which is closely related to the Minkowski sum of the domains bounded by them, can be computed with the help of support functions. Based on the approximation of the support functions of the given objects we formulate two strategies for computing convolutions and Minkowski sums. These strategies rely on piecewise approximations and decomposition into elementary domains, respectively.

## §1. Introduction

Minkowski sums of planar and spatial domains have various applications in diverse fields such as in computer aided design, image processing, computer graphics, robotics (path planning), NC machining, geometrical optics, etc. In particular, the operation of offsetting is a special case, where one of the domains is a ball. Due to space limitations, we refrain from providing a list of related references, referring instead to the introduction of [3] which provides a detailed discussion with many related references.

Convolutions of surfaces are closely related to Minkowski sums; the boundary of the Minkowski sum is contained in the convolution of the boundaries [8, 9, 12]. Consequently, the analysis of convolutions is needed in order to derive a boundary representation of Minkowski sums.

In order to deal with offsets and convolutions, the present paper studies the representation of curves and surfaces by their support functions, which is a classical tools in the field of convex geometry [2, 4, 5]. In the realm of Computer–Aided Design, this representation has been introduced in [11].

The support function of a surface assigns to each unit normal of a tangent plane the distance between that plane and the origin. The surface can be recovered from its support function by computing the envelope of the tangent planes. Consequently, the support function leads to a special instance of a dual representation (cf. [6, 10]).

Conference Title
Editors pp. 1–6.


ISBN 0-0-9728482-x-x

The remainder of this paper consists of two parts. Firstly we discuss convolutions of oriented sets, support functions and the envelope operator. Secondly we describe two strategies for computing Minkowski sums and convolutions, by local approximation and by decomposition into elementary domains.

## §2. Quasi–convex Oriented Sets and the Envelope Operator

We discuss the notions of Minkowski sums and convolutions and introduce the support function representation of quasi–convex oriented hypersurfaces.

### 2.1. The linear space of quasi–convex oriented sets

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^{n+1}$, where we consider the cases $n = 1$ and $n = 2$, is the set

$$A \oplus B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}. \tag{1}$$

Non–empty subsets $\mathcal{A}, \mathcal{B}$ of the Cartesian product $\mathbb{S} \times \mathbb{R}^{n+1}$, where $\mathbb{S} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the unit sphere, will be called *oriented sets*. They can be seen as sets of points with associated unit vectors. Let $\mathcal{O}_n$ be the set of all oriented sets. We denote with $\pi_1 : \mathcal{O}_n \to \mathbb{S}$ and $\pi_2 : \mathcal{O}_n \to \mathbb{R}^{n+1}$ the projections to the first and second component, respectively. For $\mathcal{A}, \mathcal{B} \in \mathcal{O}_n$ and $\alpha \in \mathbb{R}$ we define the *convolution*

$$\star : (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \star \mathcal{B} = \{(\mathbf{n}, \mathbf{a} + \mathbf{b}) : (\mathbf{n}, \mathbf{a}) \in \mathcal{A}, (\mathbf{n}, \mathbf{b}) \in \mathcal{B}\}. \tag{2}$$

and the *scaling*

$$\cdot : (\alpha, \mathcal{A}) \mapsto \alpha \cdot \mathcal{A} = \{(\mathbf{n}, \alpha\mathbf{a}) : (\mathbf{n}, \mathbf{a}) \in \mathcal{A}\}. \tag{3}$$

$(\mathcal{O}_n, \star, \cdot)$ is ‘almost’ a linear space, with the zero vector $\mathbb{S} \times \mathbf{0}$, but the distributivity of multiplication

$$(\alpha + \beta) \cdot \mathcal{A} = (\alpha \cdot \mathcal{A}) \star (\beta \cdot \mathcal{A}) \tag{4}$$

does not hold in general.

An oriented set $\mathcal{A}$ will be said to be *quasi-convex* if $\pi_1 : \mathcal{A} \to \mathbb{S}$ is bijective. The set of all quasi-convex sets will be denoted by $\mathcal{Q}$. It is equal to the set of all functions $\mathbb{S} \to \mathbb{R}^{n+1}$, i.e., $\mathcal{Q} = (\mathbb{R}^{n+1})^{\mathbb{S}}$. This implies

**Lemma 1.** *Along with convolution $\star$ and scaling $\cdot$, the set $\mathcal{Q}$ is a real linear space.*

As an example we consider domains $A, B \subset \mathbb{R}^{n+1}$ bounded by smooth hypersurfaces $\partial A, \partial B$. By attaching the outward pointing normals to all points, the boundaries become oriented sets $\partial\mathcal{A}, \partial\mathcal{B}$. Alternatively, each point of the boundary can be equipped with *both* possible normals, producing oriented sets $\partial\mathcal{A}', \partial\mathcal{B}'$. Then

$$\partial(A \oplus B) \subseteq \pi_2(\partial\mathcal{A} \star \partial\mathcal{B}) \subseteq \pi_2(\partial\mathcal{A}' \star \partial\mathcal{B}'), \tag{5}$$

cf. Fig. 1. The last term represents the convolution as introduced in [12].

### 2.2. Support functions and the envelope operator

Consider a function $h \in C^1(\mathbb{S}, \mathbb{R})$, where $\mathbb{S}$ is again the unit sphere. It defines the family of hyperplanes

$$T_{\mathbf{n}} = \{\mathbf{x} : \mathbf{x} \cdot \mathbf{n} = h(\mathbf{n})\}, \quad \mathbf{n} \in \mathbb{S} \tag{6}$$

in $\mathbb{R}^{n+1}$, which possesses the envelope

$$\mathbf{x}_h \Big|_{\mathbf{n}} = h \Big|_{\mathbf{n}} \mathbf{n} + \nabla_{\mathbb{S}} h \Big|_{\mathbf{n}}, \tag{7}$$

where $\nabla_{\mathbb{S}} h$ is the embedded intrinsic gradient of $h$,

$$\nabla_{\mathbb{S}} h \Big|_{\mathbf{n}} = \nabla h^* \Big|_{\mathbf{n}} - (\mathbf{n}^\top \nabla h^* \Big|_{\mathbf{n}}) \, \mathbf{n}. \tag{8}$$

which is obtained by projecting the usual gradient into the tangent hyperplane of the sphere at $\mathbf{n}$. Here, the restriction of $h^* \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$ to $\mathbb{S}$ is equal to $h$ and $\nabla$ is the usual gradient in $\mathbb{R}^{n+1}$.

**Definition 1.** *The envelope operator*

$$\mathcal{E} : \quad C^1(\mathbb{S}, \mathbb{R}) \to \mathcal{Q} : \quad h \mapsto \{(\mathbf{n}, \mathbf{x}_h(\mathbf{n})) : \mathbf{n} \in \mathbb{S}\} \tag{9}$$

*assigns to each function $h$ a quasi–convex oriented set $\mathcal{E}(h)$. Equivalently it can be seen as a mapping*

$$\mathcal{E} : \; C^1(\mathbb{S}, \mathbb{R}) \to C^0(\mathbb{S}, \mathbb{R}^{n+1}) \subset (\mathbb{R}^{n+1})^{\mathbb{S}} : \; h \mapsto \mathbf{x}_h, \tag{10}$$

*which assigns to each function $h$ a continuous function $\mathbf{x}_h : \mathbb{S} \to \mathbb{R}^{n+1}$.*

In the vicinity of regular points (see [13] for details), $\pi_2(\mathcal{E}(h))$ is a quasi–convex hypersurface, and the value of $h$ is the distance between the tangent hyperplane $T_{\mathbf{n}}$ and the origin. The function $h$ is then called the *support function* of this hypersurface. Due to

$$\|\mathbf{x}_h\|_2^2 = \|h\|_2^2 + \|\nabla_S h\|_2^2 \tag{11}$$

where $\|\cdot\|_2$ is the $L^2$ norm in $C^1(\mathbb{S}, \mathbb{R})$ and $C^0(\mathbb{S}, \mathbb{R}^{n+1})$, respectively, and

$$\|\mathbf{x}_h\|_\infty^2 \le \|h\|_\infty^2 + \|\nabla_S h\|_\infty^2, \tag{12}$$

where $\|\cdot\|_\infty$ is the $L^\infty$ norm in $C^1(\mathbb{S}, \mathbb{R})$ and $C^0(\mathbb{S}, \mathbb{R}^{n+1})$, respectively, the norm of the envelope operator can be shown to be 1, when considering the $L_2$ (resp. $L_\infty$) norm of the image space and the corresponding Sobolev norm of the domain space.

**Lemma 2.** *The envelope operator $\mathcal{E}$ defines an isomorphism between the real linear spaces*

$$(C^1(\mathbb{S},\mathbb{R}),+,\cdot) \;\cong\; (\mathcal{E}(C^1(\mathbb{S},\mathbb{R})),\star,\cdot). \qquad (13)$$

*The latter space is a subspace of $(\mathcal{Q},\star,\cdot)$.*

Indeed, the envelope operator is a linear mapping and its kernel consists solely of $\mathbb{S}\times\mathbf{0}$.

We analyze the two simplest instances of support functions: If $h(\mathbf{n}) = d$ is a constant function, then $\mathcal{E}(h)$ is the sphere with radius $|d|$ oriented by outer (if $d > 0$) or inner (if $d < 0$) normals. If $h(\mathbf{n}) = \mathbf{n}\cdot\mathbf{v}$ is a homogeneous linear function, where $\mathbf{v} \in \mathbb{R}^{n+1}$ is a constant vector, then $\mathbf{x}_h(\mathbf{n}) = \mathbf{v}$. The oriented set $\mathcal{E}(h)$ is the point $\mathbf{v}$ with attached unit normals in all directions, $\mathcal{E}(h) = \mathbb{S}\times\{\mathbf{v}\}$. The convolutions with these two objects have simple meanings: offsetting and translation, respectively. Consequently, the geometric operations of translation, rotation, scaling and offsetting of quasi–convex shapes in $\mathcal{E}(C^1(\mathbb{S},\mathbb{R}))$ correspond to the addition of a linear function, composition with rotations, multiplication by a constant, and addition of a constant, respectively.

**Remark 1.** Under certain technical regularity assumptions, which guarantee that the differential of the mapping $\mathbf{x}_h : \mathbb{S} \to \pi_2(\mathcal{E}(h))$ has maximal rank, it can be shown that the hypersurface $\pi_2(\mathcal{E}(h))$ has the same smoothness as $h$, despite the differentiation which is present in (7), see [13].

### 2.3. Parameterization of quasi-convex sets

For any given support function $h$, Eq. (7) provides a parameterization of the corresponding surface over the parameter domain $\mathbb{S}$ or any subset $U \subseteq \mathbb{S}$. On the one hand, this parameterization can be useful, e.g., for generating sample points, since a uniform sampling on $\mathbb{S}$ will provide a curvature–dependent sampling on the surface. On the other hand, by composing $\mathbf{x}_h$ with a suitable parameterization of the sphere we obtain a parameterization of the shape over an subset of $\mathbb{R}^n$.

The case of polynomials $h$ is studied in [13]. The corresponding shapes can be always parameterized by rational functions of degree $2k+2$, where $k$ is the degree of the support function. These curves and surfaces are obtained as convolutions of certain special trochoids and surfaces of revolution with trochoid profile curves, respectively.

Some interesting curves and surfaces do not possess polynomial support functions. In the case of implicitly defined curves and surfaces, which are defined as the zero set of a function $p$, the support function can be derived by eliminating $\mathbf{x}$ and $\lambda$ from the nonlinear system of equations

$$p(\mathbf{x}) = 0, \quad h - \mathbf{x}^\top\mathbf{n} = 0, \quad \mathbf{n} - \lambda\nabla p(\mathbf{x}) = 0. \qquad (14)$$

If $p$ is a polynomial, then this can be achieved with the help of resultants.

**Example 1.** We consider the ellipsoid with half-axes 2, $\sqrt{2}$ and 1 which has the support function $h = \sqrt{4{n_1}^2 + 2{n_2}^4 + {n_3}^2}$, restricted to $\mathbb{S}$. Applying the envelope operator (see (7) and (8)) gives the parameterization

$$\mathbf{x}_h = \frac{1}{\sqrt{4\,{n_1}^2 + 2\,{n_2}^2 + {n_3}^2}} \begin{pmatrix} 4n_1 \\ 2n_2 \\ n_3 \end{pmatrix}, \quad \mathbf{n} \in \mathbb{S}. \tag{15}$$

This equation can be composed with any rational parameterization of the sphere, such as

$$n_1 = \frac{1 - u^2 - v^2}{1 + u^2 + v^2}, \quad n_2 = \frac{2u}{1 + u^2 + v^2}, \quad n_3 = \frac{2v}{1 + u^2 + v^2}. \tag{16}$$

The parameterization obtained by combining (15) and (16) might be of some interest, since the offsets of the ellipsoid can be parameterized simply by adding constant multiples of $\mathbf{n} = (n_1, n_2, n_3)^\top$. More generally, if $\mathbf{x}_h$ and $\mathbf{x}_g$ are parameterizations of two surfaces, then their convolution is parameterized simply by $\mathbf{x}_{h+g} = \mathbf{x}_h + \mathbf{x}_g$, since $\mathcal{E}$ is a linear operator. The support function representation produces parameterizations of surfaces which behave nicely with respect of convolution.

**Remark 2.** Except for certain singular points, the curvature(s) of $E(h)$ can be analyzed with the help of the differential $\mathrm{d}\mathbf{x}_h$, which is the negative inverse of the Weingarten map of $E(h)$, provided that $h$ is $C^2$. Consequently, the principal curvature directions and principal curvatures can be derived from the support function $h$ and its second derivatives, cf. [13]. For instance, in the case $n = 1$ we obtain $\kappa = (h + h'')^{-1}$, where the prime indicates the differentiation with respect to the arc-length parameterization of the circle $\mathbb{S}^1$.

## §3. Computing Minkowski Sums

We apply the support function representation to the computation of Minkowski sums of planar and spatial domains. Two main problems need to be addressed. Firstly, the support function of a general shape is not known in a closed form and has to be approximated. Secondly, the support function representation cannot deal with inflections and parabolic points. Hence, the objects have to be split at these points.

We present two general methods. The first method, which is formulated in the curve case, is based on a piecewise approximation of the support function, while the second one, which is described for surfaces, represents the input domains as unions of elementary domains.

### 3.1. Using piecewise approximations

We describe two algorithms. The first algorithm converts a given planar curve into an oriented set described by a piecewise support function representation, via simultaneous $G^1$ Hermite interpolation and approximation of inner points.

**Algorithm 1.**
*Input:* A closed $G^1$ curve $\mathbf{c}$ bounding a domain $A$, a finite dimensional space of support functions $H$ defined on $\mathbb{S}^1$, along with a basis $\{\beta_j\}_{j=1}^N$.
*Output:* $\{[h_i, U_i]\}_{i=1}^m$, where $h_i \in H$, $U_i \subseteq \mathbb{S}^1$, such that the union of all curves $\mathbf{x}_{h_i}(U_i)$ is an approximation of $\partial A$ and globally $G^1$.

1. Split $\mathbf{c}$ into $m$ segments segments $\mathbf{c}_i$, $i = 1, \ldots, m$ such that
   (a) all inflection points of $\mathbf{c}$ are splitting points (i.e. none of interior points of $\mathbf{c}_i$ is an inflection), and
   (b) the unit normals along each $\mathbf{c}_i$ vary less then a prescribed constant (i.e. the Gauss image of $\mathbf{c}_i$ is sufficiently small).

   Let $\mathbf{b}_i$, $\mathbf{e}_i$ be the endpoints of $\mathbf{c}_i$.
2. Compute $U_i$ as the Gauss image of $\mathbf{c}_i$, for $i = 1, \ldots, m$. Let $\mathbf{n}_i^b$ and $\mathbf{n}_i^e$ be the end points of $U_i$.
3. For each $i$, find $h_i \in H$ which minimizes
$$\int_{U_i} (h_i(\mathbf{n}) - h_i^{\text{exact}}(\mathbf{n}))^2 \, \mathrm{d}\mathbf{n} \tag{17}$$
   and satisfies
$$\mathbf{x}_h(\mathbf{n}_i^b) = \mathbf{b}_i \quad \text{and} \quad \mathbf{x}_h(\mathbf{n}_i^e) = \mathbf{e}_i. \tag{18}$$
   where $h_i^{\text{exact}}$ is the exact support function of $\mathbf{c}_i$. After substituting the basis representation
$$h_i = \sum_{j=1}^{N} \beta_j c_{i,j} \tag{19}$$
   into (17) and (18), we obtain a quadratic objective function of the unknown coefficients $c_{i,j}$ and 4 linear constraints, and the coefficients can be computed by solving a system of linear equations. The integral (17) and its derivatives are evaluated numerically.

Error bounds can be obtained with the help of (11) and (12).

Based on the approximate representation of the given curves, the second algorithm computes the convolution of two curves.

**Algorithm 2.**
*Input:* Two oriented sets represented by the support functions with associated domains $\{[h_i, U_i]\}_{i=1}^I$, $\{[g_j, V_j]\}_{j=1}^J$.

*Output:* The convolution, again represented by support functions with associated domains $\{[f_k, W_k]\}_{k=1}^K$

1. Define $I \cdot J$ pairs of functions and sets

$$f_{i+(j-1)I} := h_i + g_j, \quad W_{i+(j-1)I} := U_i \cap V_j \tag{20}$$

2. Delete all pairs with $W_k = \emptyset$ from the list $\{[f_k, W_k]\}_{k=1}^{I \cdot J}$.

In order to compute the outermost boundary of the Minkowski sum of the two domains described by the given curves , we have to identify the outermost part of the convolution. This can be done in three steps.

1. Find all self–intersections of the convolution.
2. Identify a suitable initial point. For instance, one may choose the point with the minimum ordinate on all branches of the convolutions.
3. Trace the outermost boundary in a counterclockwise orientation, by following the current branch of the convolution until it hits a self–intersection. At self–intersections, choose the next branch whose tangent has the minimum angle to the negative tangent of the incoming branch. The tracing stops when we arrive back at the initial point.

Note that this approach does not identify inner boundaries of the Minkowski sum, which might exist. More sophisticated algorithms for analyzing the topological structure of the convolution are described in [3, 7].

**Example 2.** See Fig. 1. Here, the space $H$ of support functions consists of all polynomials in $n_1, n_2$ of maximum degree 6, restricted to the unit circle. The dimension of this space is $N = 13$, and a possible basis consists of all monomials ${n_1}^i {n_2}^j$ of degrees 5 and 6, i.e., $5 \leq i + j \leq 6$. In the first step of Algorithm 1 we split the left curve at 4 inflection points and 6 additional points into 10 pieces. The right curve, which is composed of 8 circular arcs is split into 8 segments. The support function of the left curve is approximated using the first Algorithm, while the support function of the second shape is computed directly. Note that circular arcs have linear support functions, since they can be seen as offsets of their centers. The convolution of the two curves is obtained by applying the second algorithm. It is composed of 38 segments. Finally we identified the outermost boundary of the Minkowski sum of the two domains described by the given curves.

### 3.2. Union of elementary domains

The second method exploits the fact that the operations $\oplus$ (Minkowski sum) and $\cup$ (set union) commute,

$$\left( \bigcup_i A_i \right) \oplus \left( \bigcup_j B_i \right) = \bigcup_{i,j} \left( A_i \oplus B_j \right). \tag{21}$$

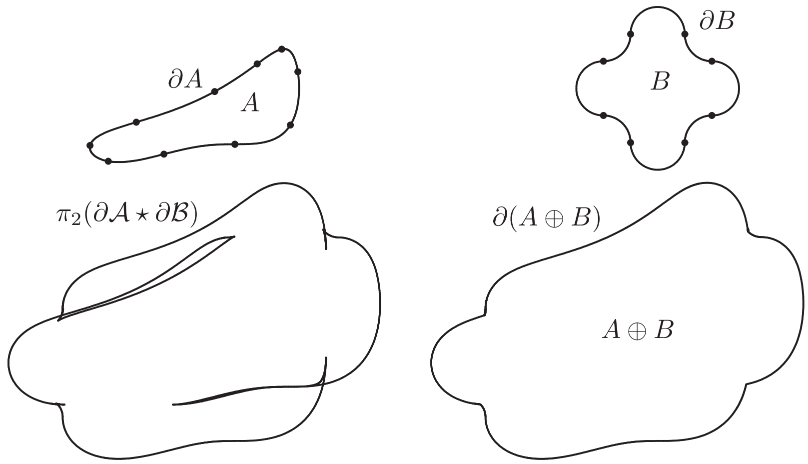


**Fig. 1.** For two given planar domains (top row) we compute the (untrimmed) oriented convolution of their boundaries (bottom left) and the boundary of the Minkowski sum of the associated planar domains (bottom right), cf. Eq. (5).

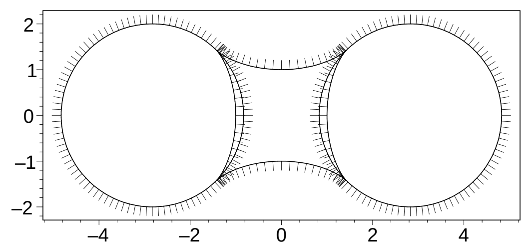


**Fig. 2.** A planar domain represented as a union of two circles and an elementary shape with a quadratic support function.

Consequently, if two domains are represented as unions of domains with known support functions, which we will call elementary domains, then their Minkowski sum can be obtained directly in the same representation.

Convex domains can be covered by convex elementary domains in a relatively simple way. For non-convex domains parts it is possible to use quasi-convex domains with non–convex segments of the boundary. Such objects always contain segments with the opposite orientation, which have to be contained in other elementary shapes, see Fig. 2.

This method seems to be particularly well suited for spatial domains, since one does not need to define a partition of $\mathbb{S}$.

**Example 3.** We computed the convolution of the boundaries of two non-convex spatial domains, which are represented as the union of three elementary domains, and a convex object without rotational symmetries.

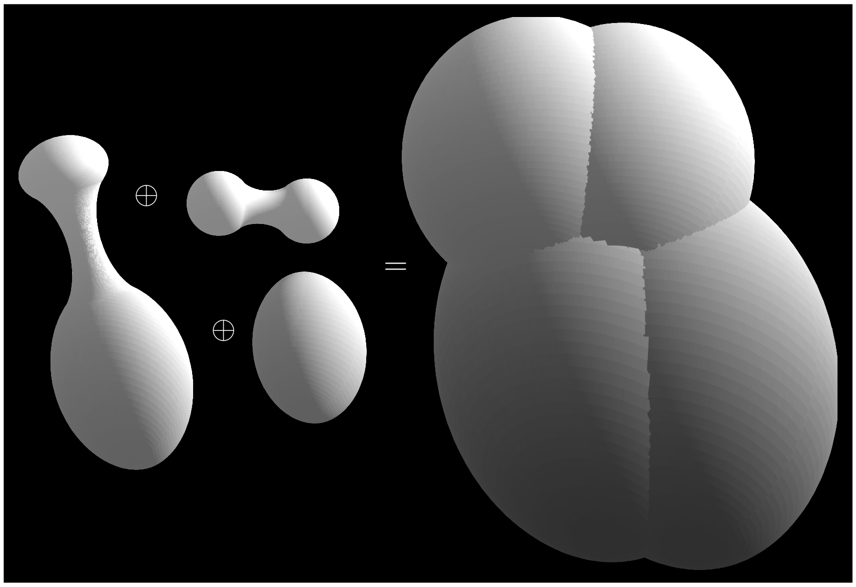

**Fig. 3.** Minkowski sum (right) of three spatial domains (left).

The domains and their Minkowski sum are shown in Fig. 3.

Similar to the curve case, the boundary of the Minkowski sum of the three domains is contained in the convolution, and it would be of some interest to identify the outermost part of the convolution, see [9]. For certain applications, such as collision detection, the representation as a union of elementary domains may even be sufficient.

## §4. Conclusion

We presented methods for computing boundaries of Minkowski sums of planar and spatial domains, which are based on the use of support functions. Suitable spaces of functions can be defined with the help of spherical splines, cf. [1]. Future work will focus on an improved treatment of inflections and parabolic points. E.g., in order to capture inflections of planar curves, support functions with unbounded second derivatives are needed, cf. Remark 2.

**Acknowledgments.** The first author was supported by a grant of the Austrian Science Fund (project no. P17387-N12) and by research project no. MSM 0021620839 at Charles University, Prague.

**References**


1. P. Alfeld, M. Neamtu and L.L. Schumaker, Fitting scattered data on sphere-like surfaces using spherical splines, *J. Comput. Appl. Math.* **73**, 5–43 (1996).
2. T. Bonnesen and W. Fenchel, *Theory of convex bodies.* BCS Ass. 1988.
3. R.T. Farouki and J. Hass, Algorithms for evaluating the boundary and covering degree of Minkowski sums and other geometrical convolutions, preprint, `www-mae.engr.ucdavis.edu/~farouki/index.html`
4. H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics* Cambridge University Press, Cambridge, 1996.
5. P.M. Gruber and J.M. Wills, *Handbook of convex geometry*, North–Holland, Amsterdam, 1993.
6. J. Hoschek, Dual Bézier curves and surfaces, in: *Surfaces in Computer Aided Geometric Design*, R.E. Barnhill and W. Boehm, eds., North Holland, 1983, 147–156.
7. A. Kaul and R.T. Farouki, Computing Minkowski sums of plane curves, *Int. J. Comput. Geom. Appl.* **5** (1995), 413-432.
8. H. Mühltaler and H. Pottmann, Computing the Minkowski sum of ruled surfaces, *Graphical Models* **65**, 369–384.
9. M. Peternell, H. Pottmann and T. Steiner, Minkowski sum boundary surfaces of 3D-objects. Technical Report no. 140 (2005), Geometry Preprint Series, Vienna Univ. of Technology.
10. H. Pottmann, Rational curves and surfaces with rational offsets, *Computer Aided Geometric Design* **12**, 175–192 (1995).
11. M. Sabin, A Class of Surfaces Closed under Five Important Geometric Operations, Technical report no. VTO/MS/207, 1974, available at `www.damtp.cam.ac.uk/user/na/people/Malcolm/vtoms/vtos.html`.
12. M.L. Sampoli, M. Peternell and B. Jüttler, Rational surfaces with linear normals and their convolutions with rational surfaces, *Comput. Aided Geom. Design* **23** (2006), 179–192.
13. Z. Šír, J. Gravesen and B. Jüttler, Curves and surfaces represented by polynomial support functions, submitted, preprint available at `www.ag.jku.at/jue_pub_en.html`.



Zbyněk Šír and Bert Jüttler
Johannes Kepler University, Institute of Applied Geometry
{zbynek.sir,bert.juettler}@jku.at, www.ag.jku.at

Jens Gravesen
Technical University of Denmark, Dept. of Mathematics
j.gravesen@mat.dtu.dk