Robust Fitting of Parametric Curves

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We consider the problem of fitting a parametric curve to a given point cloud (e.g., measurement data). Least-squares approximation, i.e., minimization of the ℓ_2 norm of residuals (the Euclidean distances to the data points), is the most common approach. This is due to its computational simplicity [1]. However, in the case of data that is affected by noise or contains outliers, this is not always the best choice, and other error functions, such as general ℓ_p norms have been considered [2]. We describe an extension of the least-squares approach which leads to Gauss-Newton-type methods for minimizing other, more general functions of the residuals, without increasing the computational costs significantly.

1 Introduction

Fitting a parametric curve or surface to a given data set is a fundamental problem in many fields in research and engineering. Most approaches are based on least-squares approximation, which minimizes the squares of (possibly orthogonal) distances from the data points to the curve resp. surface [1, 3, 4, 5]. This technique is particularly well suited for parametric curves and surface that depend linearly on their shape parameters (control points), as it is the case for polynomial spline curves and surfaces. In order to deal with orthogonal distance regression, several variants of Gauss-Newton-type techniques have been proposed in order to address the non-linearity of the problem [6].

However, the family of least-squares methods requires a fundamental assumption on the data: The errors in the data shall follow a Gaussian distribution. In many applications, this assumption cannot be guaranteed. For instance, in the presence of outliers, the use of least-squares approximation is not justified, and the minimization of other functions of the residuals is preferable. In robust statistics, this is often achieved via iteratively re-weighted least-squares (IRLS) [7]. In this variant of the ordinary least-squares method, each summand is weighted with a residual-dependent coefficient.

2 A Gauss-Newton-type approach to curve fitting

Assume that a sequence of points $\{\mathbf{P}_j\}_{j=1..N}$ with associated parameter values t_j (e.g., equidistant or chordal parameterization, see [1]) is given. We want to fit a curve $\mathbf{c}_\mathbf{s}(t)$ to the data. The parameter $t \in I = [a, b] \subset \mathbb{R}$ shall be the curve parameter and the vector $\mathbf{s}$ shall be the union of all shape parameters that describe the curve. In the case of spline curves [1], this vector contains all control points and possibly even the inner knots describing the curve. We consider the following *generalized fitting problem:*

$$F(\mathbf{s}) = \sum_{j=1}^{M} N(\|\mathbf{c}_\mathbf{s}(t_j) - \mathbf{P}_j\|) \to \min_{\mathbf{s}} . \tag{1}$$

This generalizes the least-squares fit, where $N(x) = x^2$. The class of admissible functions N is described in the following

Definition. A $\mathcal{C}^2$ function $N(x) : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *norm-like* if there exists $\epsilon \in \mathbb{R}^+$ such that the derivative satisfies

$$N'(x) = xw(x) \quad \text{for} \quad x \in (0, \epsilon] \tag{2}$$

where the associated *weight function* $w(x)$ is positive. If the weight function $w(x)$ can smoothly be extended such that $w : [0, \varepsilon] \to [c, C]$ with $c, C \in \mathbb{R}^+$, then we will call it *positive* and *bounded*.

Due to the non-linear nature of (1), iterative techniques such as Newton's method have to be used. We propose a Gauss-Newton-type approach where the exact Hessian H_F of Eq. (1) is replaced by an approximation H_F^*. This leads to the system

$$H_F^* \Delta\mathbf{s} = \nabla F \tag{3}$$

which is solved to compute the update of the shape parameters $\mathbf{s} \to \mathbf{s} + \Delta\mathbf{s}$. The gradient and the Hessian of (1) are

$$\nabla F = \sum_{j=1}^{M} w(\|\mathbf{R}_j\|)\mathbf{R}_j^\top \nabla \mathbf{R}_j \quad H_F = \sum_{j=1}^{M} \underbrace{\frac{w_j'}{\|\mathbf{R}_j\|} \nabla \mathbf{R}_j^\top \mathbf{R}_j \mathbf{R}_j^\top \nabla \mathbf{R}_j}_{(i)} + \underbrace{w_j \nabla \mathbf{R}_j^\top \nabla \mathbf{R}_j}_{(ii)} + \underbrace{w_j \nabla(\nabla \mathbf{R}_j^\top) \circ \mathbf{R}_j}_{(iii)}, \tag{4}$$

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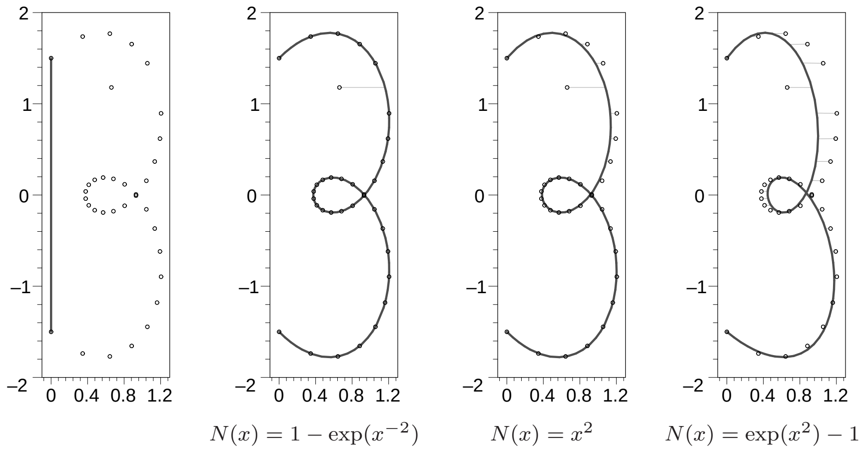

Fig. 1 A '3'-shaped point cloud was sampled from a parametric curve. After introducing an artificial error at one point, we approximated the points using three different norm-like functions (shown below). The first and the last norm-like function serve as an approximate ℓ_1 and ℓ_∞ fit. The second one is the exact ℓ_2 fit. The initial curve (leftmost plot) was chosen as a straight line. The first approximation ignores the outlier, while the third approximation yields the smallest maximum distance error.

where $\mathbf{R}_j$ denotes the residual at the current position $\mathbf{s}^c$ and where we use the abbreviations $w_j = w(\|\mathbf{R}_j\|)$ and $w'_j = w'(\|\mathbf{R}_j\|)$. The second order derivative $\nabla(\nabla\mathbf{R}_j^\top)$ is a tensor, and is to be interpreted in the following way:

$$\left[\nabla(\nabla\mathbf{R}_j^\top)\circ\mathbf{R}_j\right]_{l,k} = \sum_{i=1}^{d}\left[\frac{\partial}{\partial s_l}\frac{\partial}{\partial s_k}[\mathbf{R}_j]_i\right][\mathbf{R}_j]_i. \tag{5}$$

In order to define H_F^*, we omit the first (i) and last (iii) part in (4).

3 Discussion and examples

The system (3) of equations, which is obtained via (4) and (5), is identical to the linear system obtained from the weighted version of least-squares approximation. Each summand is multiplied with a suitable weight $w_j = w(\|\mathbf{R}_j\|)$. This weight can directly be computed from the norm-like function $N(x)$ via (2), which is evaluated at the solution of the previous iteration.

In the field of statistics, the latter approach is called iteratively re-weighted least-squares (IRLS), see [7]. The connection between IRLS and Newton's method was already established for certain special cases of fitting problems. Watson [8] investigated the use of ℓ_p norms for approximation. In [9] it is shown that a Gauss-Newton-type method for a general norm–like function of non-linear, scalar residuals leads to an IRLS problem. We extend these earlier results to the case of vector-valued residuals, which occur naturally in the case of parametric curve and surface fitting.

In the limit of zero-residual problems, the exact H_F and the approximate Hessian H_F^* can be shown to coincide [10]. Hence we obtain quadratic convergence rates in this particular case, as for the usual Gauss-Newton method for least-squares approximation. In the general case, it can still be shown that the new method produces a direction of descent. Hence, by incorporating a suitable stepsize control, the convergence towards a local minimum can be guaranteed.

An example is shown in Figure 1.

4 Conclusion

The majority of the existing curve and surface fitting methods rely on the least-squares approach which is, however, unsuitable for data contaminated by non–Gaussian noise. We generalizes this technique by replacing the usual ℓ_2 norm by a general norm-like function. Using a Gauss-Newton-type approach, the resulting method can be interpreted as an iteratively re-weighted version of the usual least-squares approximation, which is a well-known technique from robust statistics. Though presented solely for the case of curves, the ideas described in this paper can be applied to parametric surfaces, too.

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