# A Multiresolution Analysis for Tensor Product Splines Using Weighted Spline Wavelets

Mario Kapl and Bert Jüttler

*Johannes Kepler University Linz, Institute of Applied Geometry, Austria*

**Abstract**

We construct biorthogonal spline wavelets for periodic splines which extend the notion of "lazy" wavelets for linear functions (where the wavelets are simply a subset of the scaling functions) to splines of higher degree. We then use the lifting scheme in order to improve the approximation properties with respect to a norm induced by a weighted inner product with a piecewise constant weight function. Using the lifted wavelets we define a multiresolution analysis of tensor-product spline functions and apply it to image compression of black-and-white images. By performing – as a model problem – image compression with black and white images, we demonstrate that the use of a weight function allows to adapt the norm to the specific problem.

## 1 Introduction

Tensor-product spline functions are often used to describe scalar fields on subsets of $\mathbb{R}^d$, $d = 2, 3$. An important area of applications of tensor-product spline representation is Computer Aided Design [12, 13].

In the present paper we are interested in a multiresolution analysis for tensor-product spline functions. For this we will use spline wavelets, especially periodic uniform spline wavelets. Spline wavelets became a powerful mathematical tool for the hierarchical representation of geometric objects, combining the properties of splines and wavelets. Spline wavelets for tensor-products were investigated by Quak and Weyrich in [20], where the construction was described on a rectangular domain. Two relevant decomposition and reconstruction methods for tensor-product spline wavelets have already been presented in [1]. Both methods, the standard and non-standard decomposition and reconstruction are based on the one-dimensional spline wavelet transform.

Various types of spline wavelets are described in the literature. These include spline wavelets on the real line e.g. [2, 6, 7, 8, 24], (compactly supported) spline wavelets on a bounded interval e.g. [3, 5, 9, 10, 21, 22], spline wavelets with minimal support e.g. [16], and periodic spline wavelets. The latter ones are particularly useful for applications in signal processing and numerical analysis. One example of such spline wavelets is the construction of Plonka and Tasche [19] which is based on periodization of semiorthogonal Chui-Wang wavelets [6].

In the present paper we describe a non-standard tensor-product spline wavelet construction for periodic B-splines which is based on weighted spline wavelets. These are univariate spline wavelets which are adapted to a region of interest, depending on the application, by means of a weighted inner product. In the wavelet literature, such inner products have been considered in [24, 25]. In this paper we use a weighted inner product which is governed by a piecewise constant weight function with two values. We design spline wavelets such that analysis provides an approximate best approximation with respect to the norm induced by this weighted inner product.

The remainder of this paper is organized as follows. Section 2 gives an outline of the concept of spline wavelets, especially for 1-periodic uniform B-splines. Section 3 describes the construction of lazy spline wavelets, which are biorthogonal spline wavelets with a small support. Section 4 introduces the concept of weighted biorthogonal spline wavelets which are wavelets constructed from the lazy spline wavelets with the help of lifting and the weighted inner product. Section 5 describes tensor-product spline wavelets which are obtained from these weighted biorthogonal spline wavelets. We use the tensor-product spline wavelet construction for an application and compare it with standard uniform ones. Finally we conclude this paper.

## 2 Preliminaries

We give an outline of the concept of wavelets and recall several definitions concerning periodic B-splines.

### *2.1 Spline wavelets*

We follow the notation in [23]. Consider a sequence $(V^{i,d})_{i=0,1,\ldots}$ which is nested, $V^{i,d} \subset V^{i+1,d}$, of spline spaces of degree $d \in \mathbb{N}_0$. Let $(W^{i,d})_{i=0,1,\ldots}$ be a sequence of spline wavelet spaces such that $V^{j+1,d} = V^{j,d} \oplus W^{j,d}$ for all $j \in \mathbb{N}_0$. Finally let

$$\Phi^{j,d} = [\phi_0^{j,d}, \ldots, \phi_{\dim V^{j,d}-1}^{j,d}] \quad \text{and} \quad \Psi^{j,d} = [\psi_0^{j,d}, \ldots, \psi_{\dim W^{j,d}-1}^{j,d}] \tag{1}$$

be a basis of $V^{j,d}$ and $W^{j,d}$, respectively. The functions $\phi_i^{j,d}$ are called *scaling functions* and the functions $\psi_i^{j,d}$ are called *wavelets*. For ease of notation we use row vectors of functions.

Since $V^{j-1,d}$ and $W^{j-1,d}$ are subsets of $V^{j,d}$, there exist constant matrices $P^j$ and $Q^j$ such that $\Phi^{j-1,d} = \Phi^{j,d}P^j$ and $\Psi^{j-1,d} = \Phi^{j,d}Q^j$. These relations can also be expressed by a single equation, using block matrix notation

$$[\Phi^{j-1,d}|\Psi^{j-1,d}] = \Phi^{j,d}[P^j|Q^j]. \tag{2}$$

This equation is referred to as a *two-scale relation* for scaling functions and wavelets (cf. [23]). The matrices $P^j$ can be obtained with the help of knot insertion. The matrices $Q^j$ will be constructed later.

The relation between $\mathbf{c}^j$ and $\mathbf{c}^{j-1}$, $\mathbf{d}^{j-1}$ is expressed by

$$\mathbf{c}^j = [P^j|Q^j][\frac{\mathbf{c}^{j-1}}{\mathbf{d}^{j-1}}],\ \mathbf{c}^{j-1} = A^j\mathbf{c}^j,\ \mathbf{d}^{j-1} = B^j\mathbf{c}^j,\ \text{with } [\frac{A^j}{B^j}] = [P^j|Q^j]^{-1}. \tag{3}$$

Computing $\mathbf{c}^j$ from $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ is called *synthesis*. The process of splitting the coefficients $\mathbf{c}^j$ into coefficients $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ is called *analysis*. The matrices $P^j, Q^j$ and $A^j, B^j$ are called *synthesis matrices* and *analysis matrices*, respectively.

The different types of wavelets are distinguished by whether or not scaling functions and wavelets satisfy certain orthogonality relations. Let $\langle\cdot|\cdot\rangle$ be the standard inner product $\langle f|g\rangle := \int_0^1 f(x)\cdot g(x)dx$. The functions $\psi_i^{j,d}$ are called *orthogonal wavelets* if

$$\langle\phi_k^{j,d}|\phi_l^{j,d}\rangle = \delta_{kl}, \quad \langle\psi_k^{j,d}|\psi_l^{j,d}\rangle = \delta_{kl} \quad \text{and} \quad \langle\phi_k^{j,d}|\psi_l^{j,d}\rangle = 0 \tag{4}$$

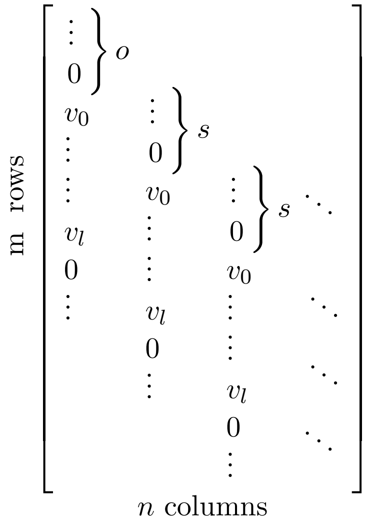
for all $j, k$ and $l$. If the orthogonality relation

$$\langle\phi_k^{j,d}|\psi_l^{j,d}\rangle = 0 \tag{5}$$

for all $j, k$ and $l$ is satisfied, we denote the wavelets as *semiorthogonal*. Otherwise we call them *biorthogonal wavelets*. We are interested in biorthogonal wavelets for 1-periodic uniform $B$-splines.

### *2.2 Periodic B-splines*

We consider 1-periodic uniform B-splines of degree $d \in \mathbb{N}_0$. Let $V^{j,d}$ be the nested spaces spanned by the 1-periodic B-splines which are constructed from




$$\mathrm{PBM}(6,3,1,2,[1,2]) = \begin{bmatrix} 0&0&2\\ 1&0&0\\ 2&0&0\\ 0&1&0\\ 0&2&0\\ 0&0&1 \end{bmatrix}$$

$$\mathrm{PBM}(6,3,-1,2,[1,2]) = \begin{bmatrix} 2&0&0\\ 0&1&0\\ 0&2&0\\ 0&0&1\\ 0&0&2\\ 1&0&0 \end{bmatrix}$$

Fig. 1. The abbreviation $\mathrm{PBM}(m, n, o, s, [v_0, \ldots, v_l])$ and two examples.

the knot sequence

$$(t^j_0, \ldots, t^j_{2^j(d+1)-1}) = \frac{1}{2^j(d+1)}(0, 1, 2, \ldots, 2^j(d+1) - 1), \tag{6}$$

where $\dim V^{j,d} = 2^j(d+1)$. The scaling functions $\phi^{j,d}_i$ are chosen as B-spline with support $[\frac{i}{2^j(d+1)}, \frac{i+d+1}{2^j(d+1)}]$ (periodically extended). In order to simplify the notations we shall use the following abbreviation.

**Definition 1** *Let $m, n, s, l \in \mathbb{N}_0$ and $o \in \mathbb{Z}$ such that $m = sn$. Let $v = [v_0, v_1, \ldots, v_l] \in \mathbb{R}^{l+1}$. We denote by $\mathrm{PBM}(m, n, o, s, [v_0, \ldots, v_l])$ the $m \times n$* ***periodic band matrix*** *$[a_{i,j}]^{j=0,\ldots,n-1}_{i=0,\ldots,m-1}$ with* ***offset*** *$o$,* ***shift*** *$s$ and* ***generic column*** *$[v_0, \ldots, v_l]$ with elements*

$$a_{(o+i+k\cdot s) \bmod m,k} = \begin{cases} v_i & \textit{if} \quad i \in \{0, \ldots, l\} \quad \textit{and} \quad k \in \{0, 1, \ldots, n-1\} \\ 0 & \textit{otherwise} \end{cases} . \tag{7}$$

See Fig. 1 for an illustration and two examples. The refinement matrix $P^j$ for 1-periodic uniform B-splines is the periodic band matrix

$$P^j = \frac{1}{2^d}\mathrm{PBM}(2^j(d+1), 2^{j-1}(d+1), 0, 2, [\tbinom{d+1}{0}, \tbinom{d+1}{1}, \ldots, \tbinom{d+1}{d+1}]). \tag{8}$$

### 2.3 *A weighted inner product*

Let $D^j \subset [0,1]$ and let $w^j : [0,1] \to \mathbb{R}$ such that

$$w^j(x) = \begin{cases} 1 & \text{for } x \in [0,1] \setminus D^j \\ u & \text{for } x \in D^j \end{cases} \tag{9}$$

where $u \in \mathbb{R}$ with $u > 0$. We use this piecewise constant function to define the weighted inner product $\langle \cdot | \cdot \rangle_{w^j}$ by

$$\langle f | g \rangle_{w^j} = \int_0^1 w^j(x) \cdot f(x) \cdot g(x) \, \mathrm{d}x. \tag{10}$$

In our case the weighted inner product $\langle \cdot | \cdot \rangle_{w^j}$ is induced by a simple non-constant weight function $w^j$. By choosing a value $u > 1$ (or $u < 1$) we emphasize the region of interest $D^j$ (or $[0,1] \setminus D^j$).

In the application described in Section 5 we specify $D^j$ as a union of intervals with the knots $t_i^j$ as end points. The choice of the length of the intervals may differ from application to application and depends on the level $j$, too.

By aligning $D^j$ with knot segments we guarantee that only a small number of different weighted spline wavelets have to be considered. All other weighted wavelets can be constructed from them with the help of translation and scaling.

## 3 Lazy spline wavelets

We describe a method for constructing biorthogonal wavelets for 1-periodic uniform B-splines of any degree $d \in \mathbb{N}$. We obtain spline wavelets with small support with banded analysis and synthesis matrices.

### 3.1 *Lazy spline wavelets*

For degree $d = 1$, one gets a particularly simple biorthogonal spline wavelet construction by choosing the wavelets $\psi_i^{j-1,1}$ for $W^{j-1,1}$ as $\psi_i^{j-1,1} = \phi_{2i}^{j,1}$, where $\{\phi_k^{j,1}\}_{k=0}^{2^j(d+1)-1}$ are linear B-splines (hat functions), see [23, 25]. Because the wavelets for $W^{j-1,1}$ are only a subset of the functions $\phi_i^{j,1}$ for $V^{j,1}$ and therefore nothing has to be done to compute them, these wavelets have been called *lazy* *wavelets* by Sweldens [25, 26]. The synthesis filters $P^j, Q^j$ and the analysis filters $A^j, B^j$ are the periodic band matrices

$$\begin{aligned} P^j &= \mathrm{PBM}(2^{j+1}, 2^j, 0, 2, [\tfrac{1}{2}, 1, \tfrac{1}{2}]) & Q^j &= \mathrm{PBM}(2^{j+1}, 2^j, 0, 2, [1]), \quad \text{and} \\ A^j &= \mathrm{PBM}(2^{j+1}, 2^j, 1, 2, [1])^T, & B^j &= \mathrm{PBM}(2^{j+1}, 2^j, -1, 2, [-\tfrac{1}{2}, 1, -\tfrac{1}{2}])^T. \end{aligned} \tag{11}$$

We extend this construction to higher degrees. [1] In this situation, the scaling functions are 1-periodic uniform B-splines of degree $d \in \mathbb{N}$. Let $k = d + 1$ the order of these B-splines. The matrix $P^j$ is known and has the form (8).

The construction of the remaining analysis and synthesis matrices $Q^j, A^j$ and $B^j$ is done as follows. We consider an auxiliary matrix $\bar{P}^j$ which we derive from $P^j$ by an index shift. Next we construct matrices $\bar{Q}^j, \bar{A}^j, \bar{B}^j$ such that $\bar{A}^j \cdot \bar{P}^j = I$, $\bar{A}^j \cdot \bar{Q}^j = 0$, $\bar{B}^j \cdot \bar{P}^j = 0$ and $\bar{B}^j \cdot \bar{Q}^j = I$. Finally we find the matrices $Q^j, A^j$ and $B^j$ from $\bar{Q}^j, \bar{A}^j, \bar{B}^j$ by another index shift.

We need to distinguish between odd and even degrees.

### *3.2 Lazy spline wavelets of odd degree*

First we apply the index shift to $P^j$,

$$\begin{aligned} \bar{P}^j &= \mathrm{PBM}(2^j k, 2^j k, -1, 1, [1]) \cdot P^j \\ &= \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1} k, -1, 2, [\tbinom{k}{0}, \tbinom{k}{1}, \ldots, \tbinom{k}{k}]). \end{aligned} \tag{12}$$

We choose the matrix $\bar{B}^j$ such that it satisfies $\bar{B}^j \cdot \bar{P}^j = 0$,

$$\bar{B}^j = \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1} k, 0, 2, [\tbinom{k}{0}, -\tbinom{k}{1}, \tbinom{k}{2}, \ldots, \tbinom{k}{k}])^T. \tag{13}$$

Now we construct the matrix

$$\bar{Q}^j = \mathrm{PBM}(2^j k, 2^{j-1} k, 1, 2, [c_0, c_1, \ldots, c_{k-2}]). \tag{14}$$

This matrix is to satisfy $\bar{B}^j \cdot \bar{Q}^j = I$. We compute the coefficients of $\bar{Q}^j$ by solving a system of linear equations. It suffices to consider the system which

[1] A more general method for constructing compactly supported biorthogonal wavelets has been described in [11], but it does not generalize the lazy wavelets (11) and gives matrices with slightly larger bandwidths ($Q^j$ has bandwidth $k$ instead of $k-1$).

is obtained by multiplying each row of $\bar{B}^j$ with the first column of $\bar{Q}^j$,

$$\frac{1}{2^{k-1}} \begin{bmatrix} -\binom{k}{1} & \binom{k}{2} & -\binom{k}{3} & \binom{k}{4} & \cdots & \binom{k}{k-2} & -\binom{k}{k-1} \\ 0 & \binom{k}{0} & -\binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-4} & -\binom{k}{k-3} \\ 0 & 0 & 0 & \binom{k}{0} & \cdots & \binom{k}{k-6} & -\binom{k}{k-5} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} & \binom{k}{6} & -\binom{k}{7} & \binom{k}{8} & \cdots & 0 & 0 \\ -\binom{k}{3} & \binom{k}{4} & -\binom{k}{5} & \binom{k}{6} & \cdots & \binom{k}{k} & 0 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{k-3} \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

As observed in Lemma 8, which is presented in the appendix, the $d \times d$-coefficient matrix of the system is regular, hence a unique solution $c_i$ exists, which is also independent of $j$. Finally, we choose

$$\bar{A}^j = \mathrm{PBM}(2^j k, 2^{j-1} k, 0, 2, [-c_{k-2}, c_{k-3}, -c_{k-4}, \ldots, c_1, -c_0])^T, \tag{16}$$

which implies $\bar{A}^j \cdot \bar{Q}^j = 0$. This matrix also satisfies $\bar{A}^j \cdot \bar{P}^j = I$, which can be shown to be equivalent to (15).

Finally we apply the inverse index shifts and get

$$\begin{gathered} Q^j = \mathrm{PBM}(2^j k, 2^j k, -1, 1, [1]) \cdot \bar{Q}^j, A^j = \bar{A}^j \cdot \mathrm{PBM}(2^j k, 2^j k, -1, 1, [1]) \\ \text{and} \quad B^j = \bar{B}^j \cdot \mathrm{PBM}(2^j k, 2^j k, 1, 1, [1]). \end{gathered} \tag{17}$$

satisfying $A^j \cdot P^j = I$, $A^j \cdot Q^j = 0$, $B^j \cdot P^j = 0$ and $B^j \cdot Q^j = I$.

**Example 2** For $d = 3$ we get

$$\begin{aligned} P^j &= \tfrac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 0, 2, [1, 4, 6, 4, 1]), \\ Q^j &= \tfrac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 0, 2, [4, 16, 4]), \\ A^j &= \tfrac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 1, 2, [-4, 16, -4])^T, \\ B^j &= \tfrac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, -1, 2, [1, -4, 16, -4, 1])^T. \end{aligned} \tag{18}$$

The wavelets and scaling functions are shown in Figure 2.

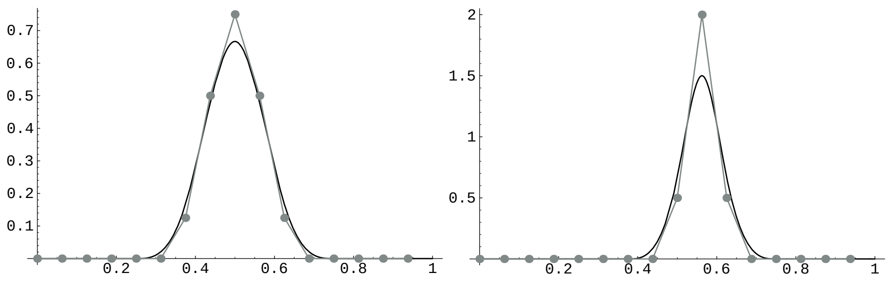



Fig. 2. 1-periodic uniform B-splines of degree $d = 3$: (left) the scaling functions $\phi_i^{j,d}$ and (right) the lazy spline wavelets $\psi_i^{j,d}$ for $j = 1$ with control points and control polygon (grey).

### *3.3 Lazy spline wavelets of even degree*

We first construct the matrices $\bar{P}^j$ and $\bar{B}^j$ as in the previous section, obtaining again (12) and

$$\bar{B}^j = \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [-\binom{k}{0}, \binom{k}{1}, -\binom{k}{2}, \ldots, \binom{k}{k}])^T. \tag{19}$$

The generic column of

$$\bar{Q}^j = \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [c_0, c_1, \ldots, c_{k-2}]), \tag{20}$$

is chosen such that $\bar{B}^j \cdot \bar{Q}^j = I$, which is equivalent to the system

$$\frac{1}{2^{k-1}} \begin{bmatrix} -\binom{k}{0} & \binom{k}{1} & -\binom{k}{2} & \cdots & -\binom{k}{k-3} & \binom{k}{k-2} \\ 0 & 0 & -\binom{k}{0} & \cdots & -\binom{k}{k-5} & \binom{k}{k-4} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{4} & \binom{k}{5} & -\binom{k}{6} & \cdots & 0 & 0 \\ -\binom{k}{2} & \binom{k}{3} & -\binom{k}{4} & \cdots & -\binom{k}{k-1} & \binom{k}{k} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{21}$$

Once again, due to Lemma 8 in the appendix, the coefficient matrix is regular, hence we obtain a unique solution which does not depend on the level $j$. Finally,

$$\bar{A}^j = \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [c_{k-2}, -c_{k-3}, c_{k-4}, \ldots, -c_0])^T \tag{22}$$

which again implies $\bar{A}^j \cdot \bar{Q}^j = 0$ and along with (21) $\bar{A}^j \cdot \bar{P}^j = I$. Finally we choose $Q^j = \bar{Q}^j$, $B^j = \bar{B}^j$ and

$$A^j = \bar{A}^j \cdot \mathrm{PBM}(2^j k, 2^j k, -2, 1, [1]). \tag{23}$$

A few simple observations are summarized below.

1. If $d$ is odd, the constructed biorthogonal spline wavelets are symmetric, otherwise they are non-symmetric.
2. The lazy spline wavelets $\psi_i^{j,d}$ have a support of the length $\frac{d}{(d+1)2^j}$.
3. An explicit formula for the coefficients $c_i$ for the lazy spline wavelets of degree $d$ can be derived with the help of Cramer's rule.

We consider scaled versions of scaling functions and wavelets (cf. [17]), i.e.

$$\hat{\Phi}^{j,d} := \sqrt{2^j(d+1)}\Phi^{j,d} \text{ and } \hat{\Psi}^{j,d} := \sqrt{2^j(d+1)}\Psi^{j,d}. \tag{24}$$

**Lemma 3** *Consider the* 1*-periodic scaled uniform* $B$*-splines* $\{\hat{\Phi}^{j,d}\}_{j\in\mathbb{N}_0}$ *of degree* $d \geq 1$ *and let* $\{\hat{\Psi}^{j,d}\}_{j\in\mathbb{N}_0}$ *be the corresponding scaled lazy spline wavelets. Then the basis* $\{\hat{\Phi}^{j,d} \cup \hat{\Psi}^{j,d}\}_{j\in\mathbb{N}_0}$ *of* $V^{j+1,d}$ *is uniformly stable.*

**Proof:** The scaled $B$-splines are uniformly stable (cf. [17]), hence there exist positive constants $M_1, M_2$ such that

$$M_1||\mathbf{c}^j||_{l^2} \leq ||\hat{\Phi}^{j,d}\mathbf{c}^j||_{L^2} \leq M_2||\mathbf{c}^j||_{l^2} \tag{25}$$

for all sequences $\mathbf{c}^j \in \mathbb{R}^{\dim V^{j,d}}$ and $j \in \mathbb{N}_0$.

Let $\hat{P}^j := \frac{1}{\sqrt{2}}P^j$ and $\hat{Q}^j := \frac{1}{\sqrt{2}}Q^j$ such that

$$[\hat{\Phi}^{j-1,d}|\hat{\Psi}^{j-1,d}] = \hat{\Phi}^{j,d}[\hat{P}^j|\hat{Q}^j]. \tag{26}$$

We denote the entries of the matrices $\hat{P}^j$ and $\hat{Q}^j$ by $\hat{p}^j_{k,l}$ and $\hat{q}^j_{k,l}$. Let

$$\begin{aligned} m_1^j &= \min\Big(\min_{k,l:\, \hat{p}^j_{k,l}\neq 0} |\hat{p}^j_{k,l}|, \min_{k,l:\, \hat{q}^j_{k,l}\neq 0} |\hat{q}^j_{k,l}|\Big), \\ m_2^j &= \max\big(\max_{k,l} |\hat{p}^j_{k,l}|, \max_{k,l} |\hat{q}^j_{k,l}|\big) \quad \text{and} \\ n^j &= \max(\text{bandwidth of } \hat{P}^j, \text{ bandwidth of } \hat{Q}^j). \end{aligned} \tag{27}$$

Due to the structure of $\hat{P}^j$ and $\hat{Q}^j$ we get values $m_1^j, m_2^j, n^j \in \mathbb{R}^+$ which are independent of the level $j$. Therefore we use for further computation the constants $m_1 = m_1^j, m_2 = m_2^j$ and $n = n^j$. We choose $j \in \mathbb{N}_0$ and consider an arbitrary but fixed sequence $(\mathbf{c}^j, \mathbf{d}^j) \in \mathbb{R}^{\dim V^{j,d} + \dim W^{j,d}}$. Then

$$
\begin{aligned}
||\hat{\Phi}^{j,d}\mathbf{c}^j + \hat{\Psi}^{j,d}\mathbf{d}^j||_{L^2} &= ||\hat{\Phi}^{j+1,d}\hat{P}^{j+1}\mathbf{c}^j + \hat{\Phi}^{j+1,d}\hat{Q}^{j+1}\mathbf{d}^j||_{L^2} \\
&\leq ||\hat{\Phi}^{j+1,d}(\hat{P}^{j+1}\mathbf{c}^j + \hat{Q}^{j+1}\mathbf{d}^j)||_{L^2} \\
&\leq M_2||\hat{P}^{j+1}\mathbf{c}^j + \hat{Q}^{j+1}\mathbf{d}^j||_{l^2} \\
&\leq M_2 m_2 n||(\mathbf{c}^j, \mathbf{d}^j)||_{l^2}.
\end{aligned}
$$

On the other hand,

$$
||\hat{\Phi}^{j,d}\mathbf{c}^j + \hat{\Psi}^{j,d}\mathbf{d}^j||_{L^2} \geq M_1||\hat{P}^{j+1}\mathbf{c}^j + \hat{Q}^{j+1}\mathbf{d}^j||_{l^2} \geq M_1 m_1||(\mathbf{c}^j, \mathbf{d}^j)||_{l^2}. \quad (28)
$$

Therefore we can find positive constants $S_1, S_2$ with $S_1 = M_1 m_1$ and $S_2 = M_2 m_2 n$ such that

$$
S_1||(\mathbf{c}^j, \mathbf{d}^j)||_{l^2} \leq ||\hat{\Phi}^{j,d}\mathbf{c}^j + \hat{\Psi}^{j,d}\mathbf{d}^j||_{L^2} \leq S_2||(\mathbf{c}^j, \mathbf{d}^j)||_{l^2} \quad (29)
$$

for all sequences $(\mathbf{c}^j, \mathbf{d}^j) \in \mathbb{R}^{\dim V^{j,d} + \dim W^{j,d}}$ and for $j \in \mathbb{N}_0$. □

**Remark 4** Let $\hat{P}^j = \frac{1}{\sqrt{2}}P^j$, $\hat{Q}^j = \frac{1}{\sqrt{2}}Q^j$, $\hat{A}^j = \sqrt{2}A^j$ and $\hat{B}^j = \sqrt{2}B^j$. The matrices $\hat{Q}^j$ are a *stable completion* (cf. [4, 10]) of the matrices $\hat{P}^j$, i.e.

$$
||[\hat{P}^j|\hat{Q}^j]||, ||[\frac{\hat{A}^j}{\hat{B}^j}]|| = \mathcal{O}(1), \text{ for all } j \in \mathbb{N}_0. \quad (30)
$$

This is equivalent to the fact that the scaled lazy spline wavelets are *uniformly stable* (cf. [4, Corollary 2.1]). Numerical experiments indicate that *Riesz stability* is not to be expected.

## 4 Lifting biorthogonal wavelets

The lifting scheme provides a tool for modifying biorthogonal wavelets, see [25, 26]. We use it in order to obtain biorthogonal wavelets which are "more orthogonal" with respect to the weighted inner product.

According to Theorem 8 in [25] one may modify the synthesis and analysis matrices of an existing biorthogonal wavelet construction according to

$$
[P^j_{\text{lift}}|Q^j_{\text{lift}}] = [P^j|Q^j - P^j S^j] \text{ and } [\frac{A^j_{\text{lift}}}{B^j_{\text{lift}}}] = [\frac{A^j + S^j B^j}{B^j}], \quad (31)
$$

where $S^j$ is an arbitrary $\dim V^{j-1,d} \times \dim V^{j-1,d}$ matrix. If $S^j$ is banded, then the new analysis and synthesis matrices are still banded, but with increased bandwidths.

Depending on the choice of $S^j$ one may construct biorthogonal wavelets with different desirable properties like increased orthogonality, higher vanishing moments etc. We use it to increase the orthogonality of the lazy wavelets.

More precisely, let $\psi_k^{j,d}$ and $\tilde{\psi}_k^{j,d}$ be biorthogonal spline wavelets such that

$$V^{j,d} \oplus W^{j,d} = V^{j,d} \oplus \text{span}\{\psi_k^{j,d}\} \quad = \quad V^{j,d} \oplus \tilde{W}^{j,d} = V^{j,d} \oplus \text{span}\{\tilde{\psi}_k^{j,d}\}. \tag{32}$$

We say that the spline wavelets $\tilde{\psi}_k^{j,d}$ have an *increased orthogonality* compared with the spline wavelets $\psi_k^{j,d}$ if

$$\min_{v^j \in V^{j,d}, \tilde{w}^j \in \tilde{W}^{j,d}} \angle(v^j, \tilde{w}^j) \quad > \min_{v^j \in V^{j,d}, w^j \in W^{j,d}} \angle(v^j, w^j), \tag{33}$$

where the $\angle$ is measured with the help of the inner product. We use the lifting scheme in order to increase the orthogonality of the lazy spline wavelets with respect to the weighted inner product.

Let $\beta^j$ be a lower bound on the angle between any two vectors in $v^j, \tilde{w}^j$, and let $f_0^{j-1} \in V^{j-1}$ be the best approximation of $f^j \in V^j$, while $f^{j-1}$ is the approximation of $f^j$ generated by wavelet analysis. Then

$$\|f_0^{j-1} - f^{j-1}\| \leq \cos\beta^j \; \|f^j - f^{j-1}\|, \tag{34}$$

i.e., the deviation from the best approximation can be bounded by the constant mutiple of the approximation error. In the orthogonal case ($\beta^j = \pi/2$), this implies $f_0^{j-1} = f^{j-1}$.

If the lifted wavelets were perfectly orthogonal, one would have that

$$[\langle \Phi^{j-1,d} | \Psi_{\text{lift}}^{j-1,d} \rangle_{w^j}] = 0, \tag{35}$$

where $\Psi_{\text{lift}}^{j-1,d} = \Phi^{j,d} Q_{\text{lift}}^j$. The system of linear equations (35) is generally overdetermined. Instead we find an approximate solution for $S^j$ by minimizing the double-sum of the squared errors $\langle \phi_i^{j-1,d}, \psi_{k,\text{lift}}^{j-1,d} \rangle^2$,

$$S^j = \arg\min_{S^j} \sum_{i=0}^{\dim V^{j-1,d}-1} \sum_{k=0}^{\dim W^{j-1,d}-1} \langle \phi_i^{j-1,d} | \psi_{k,\text{lift}}^{j-1,d} \rangle_{w^j}^2. \tag{36}$$

Since the values of one column of $S^j$ have an effect on exactly one wavelet $\psi_{k,\text{lift}}^{j-1,d}$ we can also compute the matrix $S^j$ by solving the following minimization

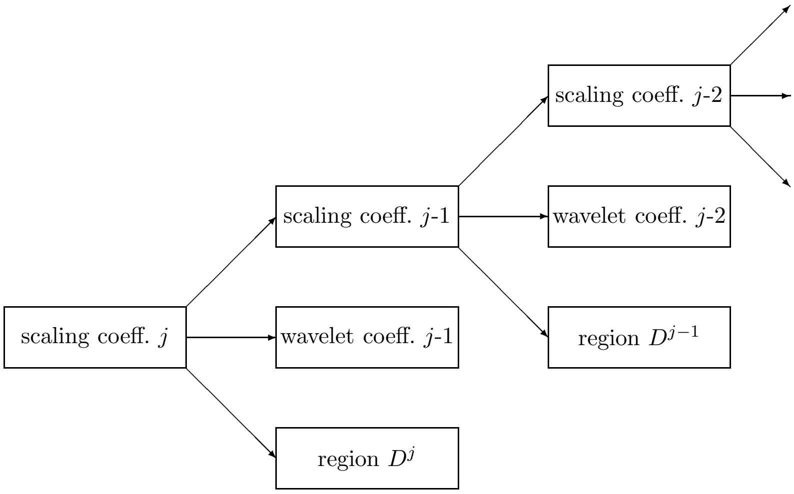



Fig. 3. The one-dimensional weighted wavelet transform

problems

$$s_k^j = \arg\min_{s_k^j} \sum_{i=0}^{\dim V^{j-1,d}-1} \langle \phi_i^{j-1,d} | \psi_{k,\text{lift}}^{j-1,d} \rangle_{w^j}^2 \tag{37}$$

for each $k \in \{0, \ldots, \dim W^{j-1,d} - 1\}$ where $s_k^j$ is the $(k+1)$-th column of $S^j$. Depending on the choice of the biorthogonal wavelet construction and the region $D^j$, we get a constant number of different minimization problems (37).

We will denote the spline wavelets constructed with the help of lifting and a standard inner product from the lazy spline wavelets as *weighted spline wavelets*. If the weight functions is constant, i.e., if the usual $L^2$ inner product is used, then we will refer to them as *standard lifted spline wavelets*. If the region $D^j$ and the support of a particular wavelet are mutually disjoint, then this wavelet is the standard lifted wavelet.

If the analysis matrices $A^j, B^j$ and synthesis matrices $P^j, Q^j$ are banded, then the lifted matrices $A^j_{\text{lift}}$,$B^j_{\text{lift}}$,$P^j_{\text{lift}}$,$Q^j_{\text{lift}}$ are still banded, but with increased bandwidths. The inner products in (36) and (37) are evaluated with the help of numerical integration. One may use Gaussian quadratures in order to obtain exact results.

Fig. 3 visualizes the one-dimensional wavelet transform using weighted spline wavelets. At each level, the information about the region of interest $D^j$ has to be kept too, since it is needed for generating the synthesis matrices.

**Example 5** Various weighted biorthogonal wavelets for 1-periodic uniform B-splines of degree 3 are visualized in Fig. 4. We constructed them by lifting

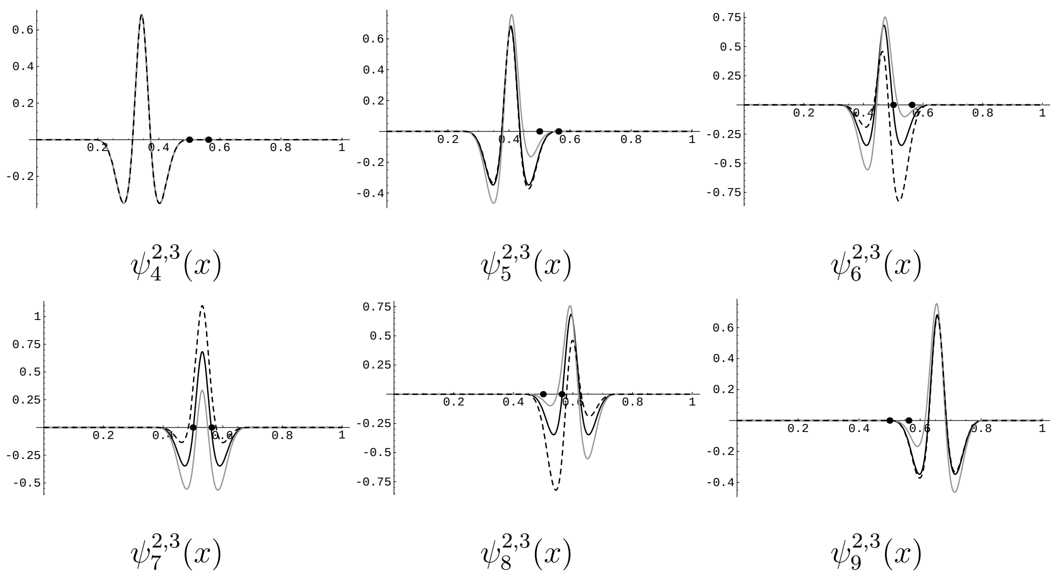



Fig. 4. Various weighted biorthogonal wavelets $\psi_i^{j-1,d}$ for 1-periodic uniform B-splines of degree $d = 3$ for $j = 3$ for $u = 0.1$ (dashed), $u = 1$ (black) and $u = 10$ (grey).The two dots mark the boundaries of $D^j$. For $u = 0.1$ and $u = 10$, only 5 wavelets differ from the standard lifted wavelet (u=1).

Table 1
Minimum angle between the spaces $V^{4,3}$ and $\tilde{W}^{4,3}$ for different values of the bandwidth $b$.

| lazy ($b = 0$) | $b = 2$ | $b = 4$ | $b = 6$ |
|---|---|---|---|
| 4.4° | 29.6° | 55.1° | 69.8° |

lazy spline wavelets using a band matrix $S^j$ with bandwidth 2. The region $D^j$ is chosen as $[\frac{4 \cdot 2^{j-2}}{4 \cdot 2^{j-1}}, \frac{4 \cdot 2^{j-2}+1}{4 \cdot 2^{j-1}}]$. The support of the lifted spline wavelets has the length $\frac{10}{4 \cdot 2^j}$.

**Example 6** Table 1 compares the orthogonality of different spline wavelet spaces $\tilde{W}^{j,3}$ for different bandwidths of the matrix $S^j$. We have computed numerically the minimal possible angle between functions of the function space $V^{j,3}$ and the wavelet spaces $\tilde{W}^{j,3}$ for lazy spline wavelets of degree $d = 3$ and standard lifted spline wavelets with bandwidths 2, 4 and 6. The orthogonality of the standard lifted spline wavelets increases with the bandwidths $b$.

## 5 Image compression

We demonstrate the possibility of adapting the region of interest to the problem by performing image compression with weighted spline wavelets. We consider a black-and-white image $\mathcal{P}$ which is represented by pixels $p_{i,j}$ with values

0 for white and 1 for black. In order to compress this image, we represent it as a tensor-product spline function and apply the so-called non-standard tensor-product wavelet transform.

Let $m, n, d \in \mathbb{N}_0$. A *tensor-product spline function* $f^{(m,n)}$ of bi-degree $(d, d)$ which is both 1-periodic with respect to $x$ and $y$ is defined by

$$f^{(m,n)} = \sum_{k=0}^{\dim V^{m,d}-1} \sum_{l=0}^{\dim V^{n,d}-1} c_{k,l}^{(m,n)} \phi_k^{m,d}(x) \phi_l^{n,d}(y), \tag{38}$$

with coefficients $c_{k,l}^{(m,n)} \in \mathbb{R}$. The upper indices $(m, n)$ refer to the level of detail of representation. The coefficients $c_{k,l}^{(m,n)} \in \mathbb{R}$ form a matrix with dimensions $(\dim V^{m,d} - 1) \times (\dim V^{n,d} - 1)$. We consider functions with uniform dyadic knots.

We use the so-called *non-standard decomposition* [1, 23] for constructing spline wavelet constructions for tensor-products. We apply alternately one analysis step of the one-dimensional (weighted) wavelet transform (see Fig. 3) to all rows and all columns of our coefficient matrix. Starting with a tensor-product function $f^{(m,m)}$, we obtain a hierarchical sequence $f^{(m,m)}$, $f^{(m-1,m)}$, $f^{(m-1,m-1)}$, $f^{(m-2,m-1)}, \ldots, f^{(0,0)}$ of functions.

More precisely, using weighted spline wavelets, the function $f^{(m,m)}$ is decomposed as

$$f^{(m,m)}(x, y) = f^{(m-1,m)}(x, y) + g^{(m-1,m)}(x, y), \tag{39}$$

where $g^{(m-1,m)}$ is a tensor-product spline function of bi-degree $(d, d)$ of the following form:

$$g^{(m-1,m)}(x, y) = \sum_{k=0}^{\dim V^{m-1,d}-1} \sum_{l=0}^{\dim V^{m,d}-1} d_{k,l}^{(m-1,m)} \psi_{k,l}^{m-1,d}(x) \phi_l^{m,d}(y), \tag{40}$$

with coefficients $d_{k,l}^{(m-1,m)} \in \mathbb{R}$ which are called *wavelet coefficients*. Furthermore $\psi_{k,l}^{m-1,d}$ are the weighted spline wavelets $\psi_k^{m-1,d}$ depending on $l$. This means, that *we can have different weighted spline wavelets for each row or column of the coefficient matrix*. Therefore we can adapt the approximation power of the non-standard tensor-product spline wavelets to the region of interest. For example, in the case of implicitly defined curves we can adapt these two-dimensional wavelets to the shape of the curve (cf. [15]).

In order to compress the black and white image, we represent it as tensor-product spline function $f^{(m,m)}$ of degree (e.g.) $(3, 3)$, where we choose the coefficients $c_{i,j}^{(m,m)}$ as the values of the pixels $p_{i,j}$. Then we apply alternately one analysis step of the one-dimensional weighted wavelet transform to all rows and all columns of the coefficient matrix $[c_{i,j}^{(m,m)}]_j^i$.

The advantage of the weighted non-standard tensor-product spline wavelet construction is that for each row or column we can choose different regions $D^j$ of interest. We wish to preserve the parts of the image $\mathcal{P}$ where the color changes black to white or vice versa. We describe these parts of the image with the help of the level curve of $f^{(m,n)}$ at 0.5, i.e.

$$\{(x,y) \in [0,1]^2 | f^{(m,n)}(x,y) = 0.5\}. \tag{41}$$

In each step the regions $D^j$ are automatically chosen, as follows. Each row or column is considered as coefficient vector of a univariate spline function. Now we choose for each of this function the region $D^j$ as union of intervals with a length of $\frac{1}{4 \cdot 2^{j-1}}$ which contain the points where this function attains the value 0.5. In our implementation, these points are computed numerically by sampling. For more advanced methods we refer to the method in [18] for computing roots of a spline function.

Furthermore we choose the weight $u$ between 2.5 and 10. Numerical experiments indicated that this is a reasonable choice for the weight $u$. If $u$ is too high, then analysis may produce additional intersection points and therefore we can obtain additional white or black pixels. On the other hand if $u$ is too low, then the effect of the weighted spline wavelets is too small.

Applying these analysis steps lead to a decomposition of the coefficients $[c_{i,j}^{(m,m)}]_j^i$ into the coefficients $[c_{i,j}^{(0,0)}]_j^i$ in the coarsest level of detail and the wavelet coefficients $[d_{i,j}^{(m-1,m)}]_j^i, \ldots [d_{i,j}^{(0,0)}]_j^i$ in the different levels of detail.

Now, in order to perform image compression, we represent the image by using only some of the wavelet coefficients. That means we delete a certain percentage of the wavelet coefficients, namely the coefficients with the smallest absolute values. We replace the values of these coefficients by zero. This leads to wavelet coefficients $[\hat{d}_{i,j}^{(m-1,m)}]_j^i, \ldots, [\hat{d}^{(0,0)_{i,j}}]_j^i$ with an increased number of zeros. We use these wavelet coefficients to reconstruct the image.

The reconstruction of the image is done by applying the inverse steps of our weighted non-standard decomposition, namely by applying weighted synthesis. That means we obtain for each level of detail the corresponding coefficients $[\hat{c}_{i,j}^{(r,s)}]_j^i$ with the help of the coefficients and wavelet coefficients of one coarser level of detail and the region of interest of the level of detail $(r,s)$. These reconstruction steps are applied from the coarsest level of detail $(0,0)$ up to the finest level of detail $(m,m)$ to obtain the coefficient matrix $[\hat{c}_{i,j}^{(m,m)}]_j^i$. Finally we use thresholding to compute the pixels $\hat{p}_{i,j}$ for the resulting image $\hat{\mathcal{P}}$.

**Example 7** Fig. 5, top row, shows the weighted wavelet transform of a simple black-and-white image. The lower rows of the figure shows the result after image compression by retaining only a certain percentage of the non-zero coefficients. These figures also visualize the pixels which are added (black) or deleted (grey) from the original image. It can clearly be seen that the weighted spline wavelets (right two columns) with adaptive choice of the region of interest performs better than standard lifted wavelets (left two columns).

## 6 Conclusion

After extending the notion of "lazy wavelets" to periodic splines of higher degree, we applied lifting in order to obtain wavelets which are more orthogonal with respect to weighted inner products. Finally we demonstrated the potential of these non-uniform wavelets, which can be adapted to the specific problem, by performing – as a model problem – image compression for a simple black-and-white image.

A weighted inner product of the form (10) can also be used to generate non-uniform semiorthogonal wavelets. This is described in more detail in the PhD thesis of the first author [14], which also presents another application of weighted inner product to structure recognition.

The paper leaves several open problems which deserve further investigation. For instance, the increased orthogonality of lifted wavelets should be discussed on a more theoretically level, and the question of an optimal choice of the weight $u$ for the weight function – and of the region of interest – requires additional studies. This may be topics of future research.

**Acknowledgments.** The authors thank the referees for their constructive comments which have helped to improve the paper. They were supported by the Austrian Science Fund (FWF) through the SFB F013 "Numerical and Symbolic Scientific Computing", subproject 15.

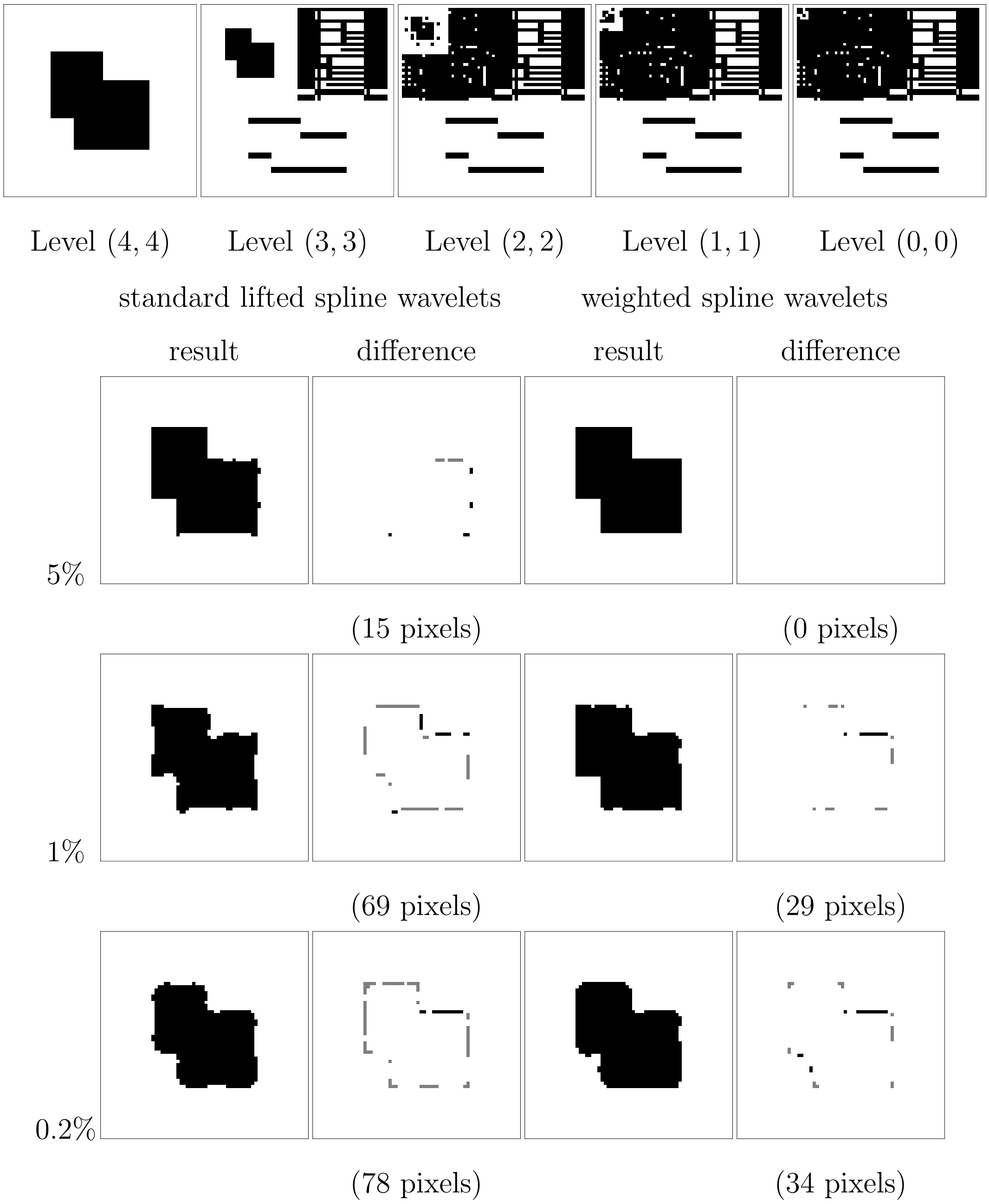



Fig. 5. Top: Weighted wavelet transform of a simple black-and-white image. Bottom: Image compression with standard lifted spline wavelets (left) and weighted spline wavelets (right) for three different compression rates.

## A Existence and uniqueness of lazy spline wavelets

**Lemma 8** Let $G^k$ be the coefficient matrix of the system of equations (15) or (21), respectively. Then

$$\det(2^{k-1}G^k) = \begin{cases} 2^{\frac{(k-1)k}{2}} & \text{if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} 2^{\frac{(k-1)k}{2}} & \text{otherwise} \end{cases} \tag{A.1}$$

**Proof:** We proceed by induction with respect to $k$. For $k = 2$ and $k = 3$ we have $\det(2G^2) = -2$ and $\det(2^2G^3) = 2^3$, respectively. For the induction step $k-1 \to k$ we will distinguish between the cases $k$ is even and $k$ is odd.

**Case 1: $k$ is even** We have

$$\det(2^{k-1}G^k) = \det \begin{bmatrix} -\binom{k}{1} & \binom{k}{2} & -\binom{k}{3} & \binom{k}{4} & \cdots & \binom{k}{k-2} & -\binom{k}{k-1} \\ 0 & \binom{k}{0} & -\binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-4} & -\binom{k}{k-3} \\ 0 & 0 & 0 & \binom{k}{0} & \cdots & \binom{k}{k-6} & -\binom{k}{k-5} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} & \binom{k}{6} & -\binom{k}{7} & \binom{k}{8} & \cdots & 0 & 0 \\ -\binom{k}{3} & \binom{k}{4} & -\binom{k}{5} & \binom{k}{6} & \cdots & \binom{k}{k} & 0 \end{bmatrix} =$$

(We add all other rows to the first row.)

$$= \det \begin{bmatrix} -2^{k-1} & 2^{k-1} & -2^{k-1} & 2^{k-1} & \cdots & 2^k & -2^k \\ 0 & \binom{k}{0} & -\binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-4} & -\binom{k}{k-3} \\ 0 & 0 & 0 & \binom{k}{0} & \cdots & \binom{k}{k-6} & -\binom{k}{k-5} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} & \binom{k}{6} & -\binom{k}{7} & \binom{k}{8} & \cdots & 0 & 0 \\ -\binom{k}{3} & \binom{k}{4} & -\binom{k}{5} & \binom{k}{6} & \cdots & \binom{k}{k} & 0 \end{bmatrix} =$$

(We add the first column to the second one, the second column to the third one, etc. and develop the result after the first row, which has only one non-zero entry left.)

$$= -2^{k-1} \det \begin{bmatrix} \binom{k}{0} & \binom{k}{0} - \binom{k}{1} & -\binom{k}{1} + \binom{k}{2} & \cdots & -\binom{k}{k-5} + \binom{k}{k-4} & \binom{k}{k-4} - \binom{k}{k-3} \\ 0 & 0 & \binom{k}{0} & \cdots & -\binom{k}{k-7} + \binom{k}{k-6} & \binom{k}{k-6} - \binom{k}{k-5} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} + \binom{k}{6} & \binom{k}{6} - \binom{k}{7} & -\binom{k}{7} + \binom{k}{8} & \cdots & 0 & 0 \\ -\binom{k}{3} + \binom{k}{4} & \binom{k}{4} - \binom{k}{5} & -\binom{k}{5} + \binom{k}{6} & \cdots & -\binom{k}{k-1} + \binom{k}{k} & \binom{k}{k} \end{bmatrix} =$$

(We multiply the first $\frac{k-2}{2}$ rows by $-1$ which gives the factor $(-1)^{\frac{k-2}{2}}$.)

$$= (-1)^{\frac{k-2}{2}}(-2^{k-1}) \det \begin{bmatrix} -\binom{k}{0} & -\binom{k}{0} + \binom{k}{1} & \binom{k}{1} - \binom{k}{2} & \cdots & \binom{k}{k-5} - \binom{k}{k-4} & -\binom{k}{k-4} + \binom{k}{k-3} \\ 0 & 0 & -\binom{k}{0} & \cdots & \binom{k}{k-7} - \binom{k}{k-6} & -\binom{k}{k-6} + \binom{k}{k-5} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} + \binom{k}{6} & \binom{k}{6} - \binom{k}{7} & -\binom{k}{7} + \binom{k}{8} & \cdots & 0 & 0 \\ -\binom{k}{3} + \binom{k}{4} & \binom{k}{4} - \binom{k}{5} & -\binom{k}{5} + \binom{k}{6} & \cdots & -\binom{k}{k-1} + \binom{k}{k} & \binom{k}{k} \end{bmatrix} =$$

(We use the binomial identity $\binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i}$...)

$$= (-1)^{\frac{k-2}{2}}(-2^{k-1}) \det \begin{bmatrix} -\binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{0} - \binom{k-1}{2} & \cdots & \binom{k-1}{k-6} - \binom{k-1}{k-4} & -\binom{k-1}{k-5} + \binom{k-1}{k-3} \\ 0 & 0 & -\binom{k-1}{0} & \cdots & \binom{k-1}{k-8} - \binom{k-1}{k-6} & -\binom{k-1}{k-7} + \binom{k-1}{k-5} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k-1}{4} + \binom{k-1}{6} & \binom{k-1}{5} - \binom{k-1}{7} & -\binom{k-1}{6} + \binom{k-1}{8} & \cdots & 0 & 0 \\ -\binom{k-1}{2} + \binom{k-1}{4} & \binom{k-1}{3} - \binom{k-1}{5} & -\binom{k-1}{4} + \binom{k-1}{6} & \cdots & -\binom{k-1}{k-2} & \binom{k-1}{k-1} \end{bmatrix} =$$

(...and add the rows 2 to $\frac{k-2}{2}$ to the first row, the rows 3 to $\frac{k-2}{2}$ to the second row, etc. Further we add the rows $k-3$ to $\frac{k}{2}$ to the $(k-2)$-th row (last row), the rows $k-4$ to $\frac{k}{2}$ to the $(k-3)$-th row, etc).

$$= (-1)^{\frac{k}{2}} 2^{k-1} \det \begin{bmatrix} -\binom{k-1}{0} & \binom{k-1}{1} & -\binom{k-1}{2} & \cdots & -\binom{k-1}{k-4} & \binom{k-1}{k-3} \\ 0 & 0 & -\binom{k-1}{0} & \cdots & -\binom{k-1}{k-4} & \binom{k-1}{k-5} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k-1}{4} & \binom{k-1}{5} & -\binom{k-1}{6} & \cdots & 0 & 0 \\ -\binom{k-1}{2} & \binom{k-1}{3} & -\binom{k-1}{4} & \cdots & -\binom{k-1}{k-2} & \binom{k-1}{k-1} \end{bmatrix} =$$

This $(k-2) \times (k-2)$-matrix is the matrix $2^{k-2}G^{k-1}$ which gives

$$= (-1)^{\frac{k}{2}} 2^{k-1} \det(2^{k-2}G^{k-1}) = (-1)^{\frac{k}{2}} 2^{k-1} 2^{\frac{(k-2)(k-1)}{2}} = (-1)^{\frac{k}{2}} 2^{\frac{(k-1)k}{2}}.$$

using the induction hypothesis.

**Case 2: $k$ is odd.** The computation is similar to the first case. Finally we obtain

$$\begin{aligned} \det(2^{k-1}G^k) &= (-1)^{\frac{k-1}{2}} 2^{k-1} \det(2^{k-2}G^{k-1}) \\ &= (-1)^{\frac{k-1}{2}} 2^{k-1} (-1)^{\frac{k-1}{2}} 2^{\frac{(k-2)(k-1)}{2}} = 2^{\frac{(k-1)k}{2}}. \end{aligned}$$

## References


[1] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms. I. In *Wavelets and applications (Marseille, 1989)*, volume 20 of *RMA Res. Notes Appl. Math.*, pages 368–393. Masson, Paris, 1992.

[2] K. Bittner. A new view on biorthogonal spline wavelets. Preprint, Universität Ulm, 2005.

[3] K. Bittner. Biorthogonal spline wavelets on the interval. In *Wavelets and splines: Athens 2005*, Mod. Methods Math., pages 93–104. Nashboro Press, Brentwood, TN, 2006.

[4] J. M. Carnicer, W. Dahmen, and J. M. Peña. Local decomposition of refinable spaces and wavelets. *Appl. Comput. Harmon. Anal.*, 3(2):127–153, 1996.

[5] C. K. Chui and E. Quak. Wavelets on a bounded interval. In *Numerical methods in approximation theory, Vol. 9 (Oberwolfach, 1991)*, Internat. Ser. Numer. Math. 105, pages 53–75. Birkhäuser, Basel, 1992.

[6] C. K. Chui and J. Wang. On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.*, 330(2):903–915, 1992.

[7] C. K. Chui and J. Wang. A study of compactly supported scaling functions and wavelets. In *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*, pages 121–140. A K Peters, Wellesley, MA, 1994.

[8] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 45(5):485–560, 1992.

[9] A. Cohen, I. Daubechies, and P. Vial. Wavelets on the interval and fast wavelet transforms. *Appl. Comput. Harmon. Anal.*, 1(1):54–81, 1993.

[10] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal spline wavelets on the interval—stability and moment conditions. *Appl. Comput. Harmon. Anal.*, 6(2):132–196, 1999.

[11] W. Dahmen and C. A. Micchelli. Banded matrices with banded inverses. II. Locally finite decomposition of spline spaces. *Constr. Approx.*, 9(2-3):263–281, 1993.

[12] G. Farin. *Curves and surfaces for computer-aided geometric design.* Academic Press, 1997.

[13] J. Hoschek and D. Lasser. *Fundamentals of computer aided geometric design.* A K Peters Ltd., Wellesley, MA, 1993.

[14] M. Kapl. *A multiresolution analysis for tensor-product splines using weighted inner products and its application to algebraic spline curves.* PhD thesis, Johannes Kepler University,Linz, 2008.

[15] M. Kapl and B. Jüttler. Multiresolution analysis for implicitly defined algebraic spline curves with weighted wavelets. In M. Neamtu and L. Schumaker, editors, *Approximation Theory 12: San Antonio 2007*, pages 191–200. Nashboro Press, 2008.

[16] T. Lyche and K. Mørken. Spline-wavelets of minimal support. In *Numerical methods in approximation theory, Vol. 9 (Oberwolfach, 1991)*, volume 105 of *Internat. Ser. Numer. Math.*, pages 177–194. Birkhäuser, Basel, 1992.

[17] T. Lyche, K. Mørken, and E. Quak. Theory and algorithms for nonuniform spline wavelets. In *Multivariate approximation and applications*, pages 152–187. Cambridge Univ. Press, Cambridge, 2001.

[18] K. Mørken and M. Reimers. An unconditionally convergent method for computing zeros of splines and polynomials. *Math. Comp.*, 76(258):845–865, 2007.

[19] G. Plonka and M. Tasche. On the computation of periodic spline wavelets. *Appl. Comput. Harmon. Anal.*, 2(1):1–14, 1995.

[20] E. Quak and N. Weyrich. Decomposition and reconstruction algorithms for bivariate spline wavelets on the unit square. In *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*, pages 419–428. A K Peters, Wellesley, MA, 1994.

[21] E. Quak and N. Weyrich. Decomposition and reconstruction algorithms for spline wavelets on a bounded interval. *Appl. Comput. Harmon. Anal.*, 1(3):217–231, 1994.

[22] E. Quak and N. Weyrich. Wavelets on the interval. In *Approximation*

*theory, wavelets and applications (Maratea, 1994)*, volume 454 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 247–283. Kluwer Acad. Publ., Dordrecht, 1995.

[23] E. J. Stollnitz, T. D. DeRose, and D. H. Salesin. *Wavelets For Computer Graphics: Theory and Application*. Morgan Kaufmann Publishers, Inc, first edition, 1996.

[24] W. Sweldens. Compactly supported wavelets which are biorthogonal with respect to a weighted inner product. In *Proceedings of the 14th Imacs World Congress*, 1996. avalaible at *http://netlib.bell-labs.com/who/wim/*, *Proceedings of the 14th Imacs World Congress*.

[25] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM J. Math. Anal.*, 29(2):511–546, 1997.

[26] W. Sweldens and P. Schröder. Building your own wavelets at home. In *Wavelets in Computer Graphics*, ACM SIGGRAPH Course Notes, 1996.