Spherical quadratic Bézier triangles with chord lengths parameterization and tripolar coordinates in space

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Abstract

We consider special rational triangular Bézier surfaces of degree 2 on the sphere in standard form and show that these surfaces are parameterized by chord lengths. More precisely, it is shown that the ratios of the three distances of a point to the patch vertices and the ratios of the distances of the parameter point to the three vertices of the (suitably chosen) domain triangle are identical. This observation extends an observation of Farin (2006) about rational quadratic curves representing circles to the case of surfaces. In addition, we discuss the relation to tripolar coordinates.

Key words: chord lengths parameterizations, quadratic rational Bézier patches, rational patches on the sphere, tripolar coordinates

1 Introduction

Rational curves with chord length parameterization have recently been studied by Farin (2006), Sánchez-Reyes and Fernández-Jambrina (2008) and Lü (2009). These curves are characterized by the relation

$$\forall t \in (0,1): \quad \frac{t-0}{||\mathbf{P}(t)-\mathbf{P}(0)||} = \frac{1-t}{||\mathbf{P}(1)-\mathbf{P}(t)||} \tag{1}$$

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between the value of the parameter and the chord length. This relation is similar to the close (polynomial) relationship between arc length and curve parameter which exists for Pythagorean-hodograph curves, see Farouki (2008).

Farin (2006) initiated the investigation of rational curves with chord length parameterization by observing that rational quadratic circles in standard Bézier form are parameterized by chord–length. Independently, a geometric proof of this fact has been derived in the context of circle-preserving subdivision curves by Sabin and Dodgson (2005).

As a potential advantage, the rational curves with this property provide a simple inversion formula, since the parameter value of any point on the curve can easily be computed using (1). For instance, this can be useful for implicititation of the curve or for projecting a point onto the curve (closest point computation). Moreover, these curves do not possess self-intersections.

Meanwhile, two constructions for general rational curves with this remarkable property were presented.

Sánchez-Reyes and Fernández-Jambrina (2008) studied the close relationship between curves in chord-length parameterization and bipolar coordinates. This led them to a compact explicit expression for all planar curves with rational chord term-length parametrization. In addition to straight lines and circles in standard form, this class of curves was shown to contain remarkable curves, such as the equilateral hyperbola, Bernoulli's lemniscate and Pascal's Limaçon.

Lü (2009) uses computations in the complex plane to study rational curves which can be parameterized by chord length and he presents two schemes characterizing planar and spatial curves. Among other results, the low-degree rational curves such as cubics and quartics are studied and applied to geometric Hermite interpolation.

The present paper is devoted to the equal chord property of quadratic rational Bézier patches that describe a segment of a sphere, thus extending the results of Farin (2006) to the case of surfaces. We use the well-known construction of spherical quadratic patches by stereographic projection (Teller and Séquin, 1991; Albrecht, 2004; Dietz, Hoschek and Jüttler, 1993; Farin, 2002) and show that a subset of these patches possesses a property which generalizes the relation (1) to the case of surfaces. Finally we show how to characterize this property using tripolar coordinates in space, thereby extending the observations of Sánchez-Reyes and Fernández-Jambrina (2008) concerning the relation between bipolar coordinates and curves with chord-length parameterization.

2 Equal chord lengths property of spherical quadratic patches

We discuss a special class of spherical patches and analyze their equal chord lengths property.

2.1 Special quadratic triangular patches on the sphere

We consider a rational quadratic triangular Bézier patch

$$\mathbf{P}(u,v,w) = \frac{\displaystyle\sum_{i,j,k\in\mathbb{Z}^+,\; i+j+k=2} \omega_{ijk}\,\mathbf{b}_{ijk}\,\frac{2}{i!j!k!}u^i v^j w^k}{\displaystyle\sum_{i,j,k\in\mathbb{Z}^+,\; i+j+k=2} \omega_{ijk}\,\frac{2}{i!j!k!}u^i v^j w^k}, \quad (u,v,w)\in\Delta \tag{2}$$

with the domain $\Delta = \{(u,v,w):\; u,v,w \geq 0,\; u+v+w=1\}$. Its shape is determined by the six control points $\mathbf{b}_{ijk}$ with the associated weights ω_{ijk}. In particular, we study quadratic patches which represent a segment of the sphere.

It is well known that the three boundary curves of a spherical quadratic Bézier patch intersect in a single point $\mathbf{c}$ on the sphere, which we will call the *center of projection* (Dietz, Hoschek and Jüttler, 1993; Farin, 2002). We say that a spherical patch $\mathbf{P}$ is *special*, if the line spanned by $\mathbf{c}$ and the midpoint $\mathbf{m}$ of the circumcircle $\mathcal{C}$ of the three patch vertices $\mathbf{V}_1 = \mathbf{b}_{200}$, $\mathbf{V}_2 = \mathbf{b}_{020}$, $\mathbf{V}_3 = \mathbf{b}_{002}$ intersects the plane spanned by these vertices orthogonally, see Fig. 1.

With the help of a suitable translation, rotation and scaling, any such patch can be transformed into a *canonical position*, which is characterized by the following three properties:

(1) The sphere is centered at the origin and has radius 1.
(2) The center of projection $\mathbf{c}$ is the "south pole" $\mathbf{c} = (0,0,-1)^\top$.
(3) The three vertices of the patch have the same vertical coordinate.

In addition, we may apply a rational bilinear reparameterization which transforms the triangular patch into standard form, with equal vertex weights $\omega_{200} = \omega_{020} = \omega_{002}$ (see Farin, 1999).

Recall that the inverse stereographic projection with center $\mathbf{c}$ maps any point $\mathbf{x} = (x_1, x_2, x_3)^\top$ in space into the second intersection point (different from $\mathbf{c}$) of the line spanned by $\mathbf{x}$ and $\mathbf{c}$ with the unit sphere,

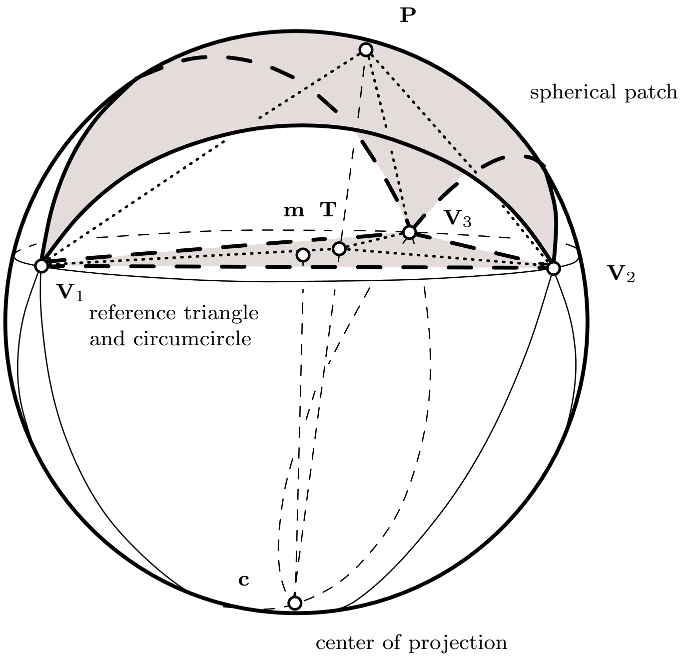

Fig. 1. A special quadratic triangular patch on the sphere with the reference triangle $\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$ and the center of projection $\mathbf{c}$. The line $\mathbf{cm}$ intersects the plane spanned by the triangle orthogonally.

$$\sigma : \mathbf{x} \mapsto \frac{1}{1 + {x_1}^2 + {x_2}^2 + {x_3}^2 + 2\,x_3} \begin{pmatrix} 2x_1(1+x_3) \\ 2x_2(1+x_3) \\ 1 - {x_1}^2 - {x_2}^2 + {x_3}^2 + 2x_3 \end{pmatrix}. \tag{3}$$

Any special spherical patch $\mathbf{P}$ in standard form and canonical position can be constructed by applying this mapping to the linear triangular Bézier patch

$$\mathbf{T}(u, v, w) = u\mathbf{V}_1 + v\mathbf{V}_2 + w\mathbf{V}_3, \quad (u, v, w) \in \Delta, \tag{4}$$

with the three vertices (control points)

$$\mathbf{V}_\ell = \left(\frac{2rc_\ell}{1+r^2}, \frac{2rs_\ell}{1+r^2}, \frac{1-r^2}{1+r^2} \right)^\top, \quad \ell = 1, 2, 3, \tag{5}$$

where the parameter $r \geq 0$ controls the common vertical coordinate. Here we use the abbreviations $c_\ell = \cos\phi_\ell$ and $s_\ell = \sin\phi_\ell$, where the three angles ϕ_ℓ specify the position of the three vertices on the circle of latitude $\mathcal{C}$ obtained by intersecting the plane $x_3 = (1 - r^2)/(1 + r^2)$ with the unit sphere.

This patch will be called the *reference triangle*. Since the inverse stereographic projection σ preserves points on the sphere, the three points $(\mathbf{V}_\ell)_{\ell=1,2,3}$ are the vertices of the patch $\mathbf{P}$.

Remark 1 After substituting the linear patch (4) into σ and a short computation, one obtains the control points and weights $\mathbf{b}_{200} = \mathbf{V}_1$, $\mathbf{b}_{020} = \mathbf{V}_2$,

$\mathbf{b}_{002} = \mathbf{V}_3$, $\omega_{200} = \omega_{020} = \omega_{002} = 1 + r^2$ and

$$\mathbf{b}_{ijk} = \frac{1}{1 + r^2 d_{\ell m}} \begin{pmatrix} r(c_\ell + c_m) \\ r(s_\ell + s_m) \\ 1 - r^2 d_{\ell m} \end{pmatrix}, \quad \omega_{ijk} = 1 + r^2 d_{\ell m}, \tag{6}$$

where $(i, j, k, \ell, m) \in \{(1, 1, 0, 1, 2), (1, 0, 1, 1, 3), (0, 1, 1, 2, 3)\}$ and

$$d_{\ell m} = c_\ell c_m + s_\ell s_m.$$

2.2 The equal chord lengths property

The following theorem extends a result of Farin (2006) to the case of surfaces.

Theorem 2 *We consider a special quadratic triangular patch* $\mathbf{P}$ *on the sphere in standard form with associated reference triangle* $\mathbf{T}$*, cf. (4). The distances*

$$r_\ell(u, v, w) = ||\mathbf{V}_\ell - \mathbf{P}(u, v, w)|| \ \text{ and } \ R_\ell(u, v, w) = ||\mathbf{V}_\ell - \mathbf{T}(u, v, w)|| \tag{7}$$

from the three vertices of the patch to the points of the surface and to the points of the reference triangle ($\ell = 1, 2, 3$*), respectively, satisfy*

$$\forall (u, v, w) \in \Delta^0 : \quad \frac{R_1}{r_1} = \frac{R_2}{r_2} = \frac{R_3}{r_3}, \tag{8}$$

where Δ^0 *is the parameter domain of the patch without its three vertices.*

Proof. Let d be the distance between the center of projection $\mathbf{c}$ and any of the vertices $\mathbf{V}_\ell$. The three distances have the same value, since the patch is assumed to be special. We consider the circumcircle $\mathcal{C}$ of the three vertices $(\mathbf{V}_\ell)_{\ell=1,2,3}$ (and hence of the reference triangle). The plane spanned by $\mathbf{V}_\ell$, $\mathbf{c}$ and $\mathbf{P}$ intersects this circle in $\mathbf{V}_\ell$ and in another point $\mathbf{V}'_\ell$, which also possesses the distance d from $\mathbf{c}$. Moreover, the lines connecting $\mathbf{V}_\ell$ with $\mathbf{V}'_\ell$ and $\mathbf{c}$ with $\mathbf{P}$ intersect at the corresponding point $\mathbf{T} = \mathbf{T}(u, v, w)$ of the reference triangle, see Fig. 2. We have $\mathbf{P} \neq \mathbf{T}$, as the the parameters (u, v, w) are contained in the domain Δ^0 without vertices.

According to the intersecting chord theorem, when two chords intersect each other inside a circle, the products of their segments are equal. Using this theorem for $\mathbf{T}$ with respect to the circumcircle $\mathcal{C}_\ell$ of $\mathbf{V}_\ell$, $\mathbf{P}$, $\mathbf{V}'_\ell$ and $\mathbf{c}$ we get

$$\|\mathbf{V}_\ell - \mathbf{T}\| \cdot \|\mathbf{V}'_\ell - \mathbf{T}\| = \|\mathbf{P} - \mathbf{T}\| \cdot \|\mathbf{c} - \mathbf{T}\|. \tag{9}$$

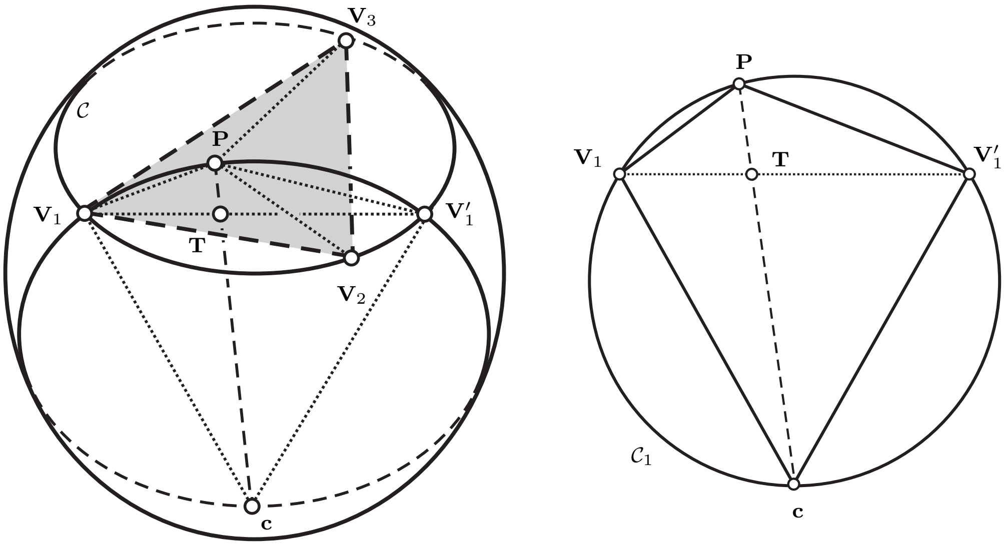

Fig. 2. The points used for the proof of Theorem 2 – the spatial situation (left) and the planar section through the points $\mathbf{c}, \mathbf{P}, \mathbf{V}_1$ (right).

Since $\angle(\mathbf{V}_\ell, \mathbf{T}, \mathbf{P}) = \angle(\mathbf{c}, \mathbf{T}, \mathbf{V}_\ell')$, we conclude that the two triangles $\triangle(\mathbf{V}_\ell, \mathbf{T}, \mathbf{P})$ and $\triangle(\mathbf{c}, \mathbf{T}, \mathbf{V}_\ell')$ are similar. Consequently,

$$\frac{R_\ell}{r_\ell} = \frac{||\mathbf{V}_\ell - \mathbf{T}||}{||\mathbf{V}_\ell - \mathbf{P}||} = \frac{||\mathbf{c} - \mathbf{T}||}{||\mathbf{V}_\ell' - \mathbf{c}||} = \frac{||\mathbf{c} - \mathbf{T}||}{d}. \tag{10}$$

This proves the assertion, since the ratio is the same for $\ell = 1, 2, 3$. $\square$

Remark 3 As demonstrated by the proof, the ratio R/r is even the same for the distances $\hat{r} = ||\hat{\mathbf{V}} - \mathbf{P}||$ and $\hat{R} = ||\hat{\mathbf{V}} - \mathbf{T}||$ to *any* point $\hat{\mathbf{V}}$ on the circumcircle $\mathcal{C}$.

As a consequence of this observation, one can conclude that the boundary curves of a special quadratic triangular patch are parameterized by chord length. This is even true for any spherical quadratic Bézier patch in standard form, not just for special ones. However, Theorem 2 cannot be extended to general quadratic patches on the sphere.

3 Bipolar and tripolar coordinates

We recall the notion of bipolar coordinates in the plane and its relation to curves with chord length parameterization. In the second part, we extend these observations to the case of surfaces.

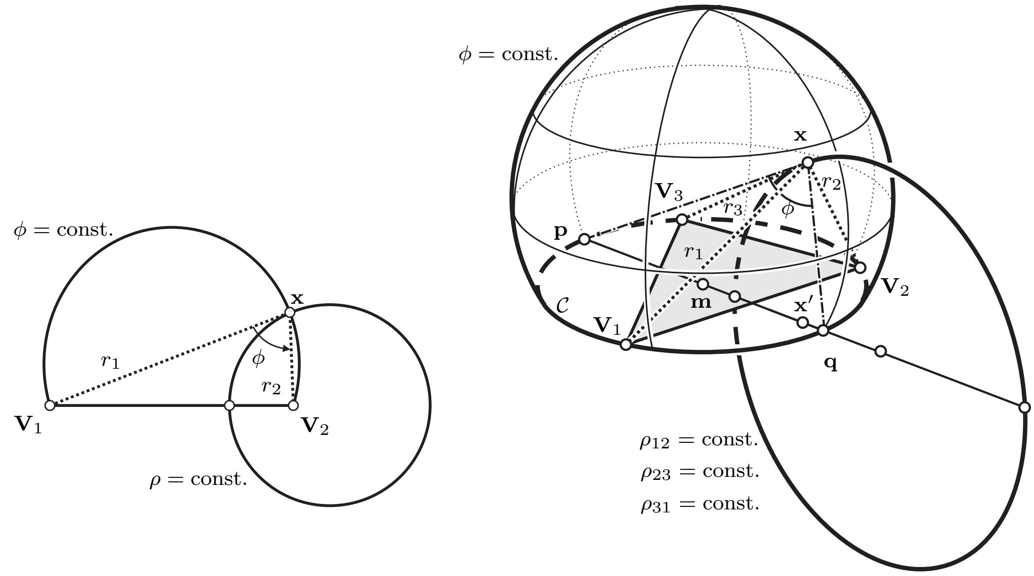

Fig. 3. Bipolar coordinates in the plane (left) and tripolar coordinates in three-dimensional space (right).

3.1 Bipolar coordinates in the plane

Recall that bipolar coordinates in the plane, which are defined with the help of the distances r_1, r_2 of any point $\mathbf{x}$ to two fixed vertices (or poles) $\mathbf{V}_1$, $\mathbf{V}_2$, are based on the following geometric facts (see Bateman, 1938; Sánchez-Reyes and Fernández-Jambrina, 2008; Farouki and Moon, 2000), see Fig. 3 (left):

(1) The set of all points with constant ratio $\rho = r_1/r_2$ is a circle (Apollonius' definition of a circle).
(2) The set of all points $\mathbf{x}$ with constant oriented angle $\phi = \angle(\mathbf{V}_1, \mathbf{x}, \mathbf{V}_2)$ is a circular arc. (This follows from the inscribed angle theorem)
(3) Any two of the circles which are obtained for constant values of ϕ or ρ intersect each other orthogonally.

Consequently, the ratio ρ, which will be called the *bipolar distance ratio*, and the angle ϕ uniquely determine the location of a point $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{V}_1, \mathbf{V}_2\}$. The ratio ρ takes its values in $\mathbb{R}^+$. The angle ϕ should be chosen in $(0, \pi)$ and in $(\pi, 2\pi)$ if $\mathbf{x}$ lies in the left and right-hand side of the oriented line from $\mathbf{V}_1$ to $\mathbf{V}_2$, respectively. Moreover, it is equal to π / to 0 for points on the line which are between / not between the two vertices.

The curves of constant ϕ or ρ are the circles through the two vertices and their orthogonal system. By applying an inversion at a circle with center $\mathbf{V}_2$ and radius $||\mathbf{V}_1 - \mathbf{V}_2||$, the two systems of isocurves are mapped to the pencil of lines through $\mathbf{V}_1$ and to concentric circles around $\mathbf{V}_1$, respectively.

Bipolar coordinates can be used to characterize curves with chord length parameterization.

Definition 4 *A curve segment* $\mathbf{P} : [t_0, t_1] \to \mathbb{R}^2$ *is said to be parameterized by chord length, if the bipolar distance ratio* ρ *of a curve point with respect to the segment end points is equal to the bipolar distance ratio of the parameter value to the boundary points* t_0, t_1 *of the parameter domain.*

3.2 Tripolar coordinates in space

The observations about special spherical quadratic patches, which were presented in the previous section, serve as the motivation for the definition of tripolar coordinates in space. Note that these coordinates are different from the concept of tripolar coordinates in the plane (see e.g. Farouki and Moon (2000); Bottema (2008)).

We consider the distances r_1, r_2, r_3 of any point $\mathbf{x}$ to three non–collinear vertices $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ (poles). Let $\mathcal{C}$ be the circumcircle of the three vertices.

Similar to the planar case, we formulate three geometric observations, see Fig. 3 (right):

(1) The set of all points with constant ratios

$$\rho_{\ell m} = r_\ell / r_m, \quad (\ell, m) \in \mathcal{I} = \{(1,2), (2,3), (3,1)\} \tag{11}$$

is a circle. Any single constant ratio $\rho_{\ell m}$ defines a sphere with an axis through $\mathbf{V}_\ell$ and $\mathbf{V}_m$ (by using again Apollonius' definition of a circle). Obviously, the three ratios are not independent. They are related by

$$\rho_{12}\rho_{23}\rho_{31} = 1. \tag{12}$$

The three spheres defined by the three ratios intersect in a single circle.

(2) For any point $\mathbf{x}$, let $\mathbf{x}'$ be its orthogonal projection into the plane containing the circumcircle $\mathcal{C}$ (see Fig. 3b). We consider a line in this plane that passes through the center $\mathbf{m}$ of the circumcircle and through $\mathbf{x}'$. It intersects the circumcircle in two points $\mathbf{p}$, $\mathbf{q}$, i.e., $\mathbf{pq}$ is a diameter of the circumcircle. We consider the angle

$$\phi(\mathbf{x}) = \angle(\mathbf{p}, \mathbf{x}, \mathbf{q}). \tag{13}$$

This angle is well defined for all points $\mathbf{x} \notin \mathcal{C}$, since the diameter $\mathbf{pq}$ is unique, except for the points where $\mathbf{x}' = \mathbf{m}$. In this situation, however, the angle is independent of the choice of the diameter.

For any given diameter $\mathbf{p}'\mathbf{q}'$ of the circumcircle, the points that possess a certain constant value of their angle $\phi(\mathbf{x})$ at this diameter form two circular arcs which intersect the circumcircle orthogonally at $\mathbf{p}'$ and $\mathbf{q}'$ (inscribed angle theorem). The two circular arcs are symmetric with respect to $\mathbf{p}'\mathbf{q}'$. The collection of these circular arcs for all diameters gives two spherical caps.

The set of all points $\mathbf{x}$ with constant angle $\phi(\mathbf{x})$ lies on two spheres, which pass through the circumcircle $\mathcal{C}$. More precisely, it consists of two spherical caps that meet along the circumcircle $\mathcal{C}$ and are symmetric with respect to the plane containing it.

(3) Any spherical cap and any of the circles which are obtained for constant values of ϕ or $(\rho_{\ell m})_{(\ell,m)\in\mathcal{I}}$, respectively, intersect each other orthogonally.

Any of the circles and spherical caps possess a unique intersection. Consequently, the ratios $\rho_{\ell m}$ and the angle ϕ uniquely determine the location of a point $\mathbf{x} \in \mathbb{R}^3 \setminus \mathcal{C}$, provided that it is known on which side of the plane containing $\mathcal{C}$ it lies.

The three ratios $\rho_{\ell m}$, which we will call the *tripolar distance ratios*, determine a point in a three-dimensional space with coordinates ρ_{12}, ρ_{23} and ρ_{31}. Let $\mathcal{R}$ be the part of the surface defined by (12) which is contained in the first octant of this space (all ratios positive). The ratios $(\rho_{12}, \rho_{23}, \rho_{31})$ take their values in $\mathcal{R}$.

The angle ϕ takes its values in $[0, \pi]$. In order to uniquely characterize the location of a point, one can redefine it using the oriented volume V of the tetrahedron with vertices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{x}$,

$$\phi' = \begin{cases} \phi & \text{if } V > 0, \\ 2\pi - \phi & \text{if } V < 0, \\ \pi & \text{if } V = 0 \text{ and } \mathbf{x} \in \operatorname{int}\mathcal{C}, \\ 0 & \text{if } V = 0 \text{ and } \mathbf{x} \notin \operatorname{int}\mathcal{C}. \end{cases} \tag{14}$$

If we identify 0 and 2π, then the angle ϕ' depends continuously on $\mathbf{x}$, except for the points on $\mathcal{C}$.

The surfaces and curves of constant ϕ' or $(\rho_{\ell m})_{(\ell,m)\in\mathcal{I}} \in \mathcal{R}$ are the spheres through the circumcircle $\mathcal{C}$ and their orthogonal system, respectively. By applying an inversion at a sphere with its center on the circumcircle and whose radius is equal to the diameter of the circumcircle, the two systems of isosurfaces and isocurves are mapped to the pencil of planes through a line (the tangent of the circumcircle at the point where the inversion sphere touches it) and to coaxial circles around this line.

Since the orthogonal circles of the system of spheres through the circumcircle $\mathcal{C}$ do not depend on the choice of the inscribed triangle $\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$, we also have the following result.

Proposition 5 *For any two points $\hat{\mathbf{V}}$ and $\tilde{\mathbf{V}}$ of the circumcircle, the ratio $\hat{r}/\tilde{r}$ is constant for all points $\mathbf{x}$ on one of the circles which are characterized by the three constant ratios (11), where $\hat{r}$ and $\tilde{r}$ are the distances to $\hat{\mathbf{V}}$ and $\tilde{\mathbf{V}}$, respectively.*

Finally we use tripolar coordinates in order to characterize surface patches with a chord lengths parameterization.

Definition 6 *A triangular surface patch* $\mathbf{P} : \Delta^* \to \mathbb{R}^3$ *is said to be parameterized by chord length, if the tripolar distance ratios* $(\rho_{ij})_{(i,j)\in\mathcal{I}}$ *of a surface point with respect to the three vertices are equal to the tripolar distance ratios of the parameter point in* Δ^* *to the three vertices of the domain triangle* Δ^*.

Clearly, according to this definition, the property depends on the choice of the domain triangle. For instance, a special spherical patch does not possess this property with respect to the standard domain Δ, but only with respect to the reference triangle (or any other triangle which is similar to it). Only equilateral special spherical triangles possess this property with respect to Δ.

3.3 Computation of parameter values from distances

Consider a point $\mathbf{P}(u, v, w)$ on the given general (i.e., not only spherical) patch parameterized by chord lengths and its distances r_1, r_2, r_3 to the vertices $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ of the reference triangle $\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$. The question is how to compute the parameters $(u, v, w) \in \Delta^0$ from the values r_1, r_2, r_3.

This problem is equivalent to finding the point $\mathbf{T}$ inside $\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$ which fulfills the condition $R_1 : R_2 : R_3 = r_1 : r_2 : r_3$, where $R_\ell = \|\mathbf{V}_\ell - \mathbf{T}\|$, $\ell = 1, 2, 3$. Then we obtain

$$u = \frac{\text{area}\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{T})}{\text{area}\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)}, \quad v = \frac{\text{area}\triangle(\mathbf{V}_2, \mathbf{V}_3, \mathbf{T})}{\text{area}\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)}, \quad w = \frac{\text{area}\triangle(\mathbf{V}_3, \mathbf{V}_1, \mathbf{T})}{\text{area}\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)}, \tag{15}$$

where the areas of the triangles are computed using Heron's formula. Using the Euler identity relating $(R_\ell)_{\ell=1,2,3}$ and the sides a, b, c of the reference triangle (see Fig. 4)

$$\begin{aligned} &R_1^4a^2+R_2^4b^2+R_3^4c^2+(-a^2+c^2-b^2)R_1^2R_2^2+(-a^2+b^2-c^2)R_1^2R_3^2+\\ &\quad+(a^2-b^2-c^2)R_2^2R_3^2+(a^2-b^2-c^2)a^2R_1^2+(-a^2-c^2+b^2)b^2R_2^2+\\ &\quad\quad+(-a^2-b^2+c^2)c^2R_3^2+a^2b^2c^2 = 0 \end{aligned} \tag{16}$$

cf. Bottema (2008), and substituting $R_\ell = tr_\ell$ into (16) we arrive at a quadratic equation in t^2

$$At^4 + Bt^2 + C = 0, \tag{17}$$

where

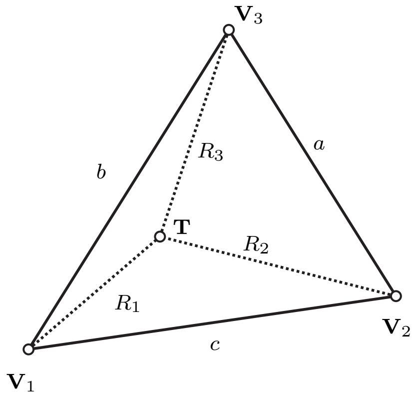

Fig. 4. The notions used in the Euler identity for tripolar coordinates in the plane.

$$
\begin{aligned}
A &= r_1^4a^2+r_2^4b^2+r_3^4c^2+r_1^2r_2^2(-a^2-b^2+c^2)+r_1^2r_3^2(-a^2+b^2-c^2)+ \\
&\quad +r_2^2r_3^2(a^2-b^2-c^2), \\
B &= r_1^2a^2(a^2-b^2-c^2)+r_2^2b^2(b^2-a^2-c^2)+r_3^2c^2(c^2-a^2-b^2), \\
C &= a^2b^2c^2.
\end{aligned}
\tag{18}
$$

The discriminant $\delta = B^2 - 4AC$ of this equation has the form

$$
\begin{aligned}
\delta = (a+b+c)(a-b-c)(a+b-c)(a-b+c)(r_1a+r_2b-r_3c)\cdot \\
\cdot (r_1a+r_2b+r_3c)(r_1a-r_2b-r_3c)(r_1a+r_3c-r_2b).
\end{aligned}
\tag{19}
$$

As a, b, c are sides of the reference triangle and thus they fulfil the triangle inequalities, the roots t^2 are real only if there exists a triangle with sides r_1a, r_2b, r_3c. Moreover, both roots t^2 are positive (which gives two positive roots for t) iff

$$
\delta > 0, \quad \text{and} \quad A > 0, B < 0, C > 0, \tag{20}
$$

for more details see Bottema (2008). It follows that under the conditions (20) we obtain two points, which are the intersections of the three Apollonius circles defined by the points $(\mathbf{V}_\ell)_{\ell=1,2,3}$ and the ratios $R_1 : R_2$, $R_2 : R_3$ and $R_3 : R_1$. Clearly, only one of these two points lies inside $\triangle(\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3)$.

4 Conclusion and future work

We identified a class of rational triangular Bézier surface patches which possess a parameterization which preserves the distance ratios to the vertices of the domain triangle. This extends the property of chord-length parameterization of quadratic rational curves, which was analyzed by Farin (2006), to the case of surfaces. In addition, we described the relation of this property to tripolar coordinates in space.

A construction for general general rational surface patches with this property is presented in the forthcoming paper Bastl et al. (201x), which also demonstrates that there exists a fairly large class of surface patches with rational chord-length parameterization.

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References

Albrecht, G. (1998), An algorithm for Parametric Quadratic Patch Construction. *Computing* **72**, 1–12.

Bastl, B., Jüttler, B., Lávička, M., Šír, Z. (201x), Surfaces with rational chord length parameterization, Proc. Geometric Modeling and Processing, Springer Lecture Notes in Computer Science vol. 6130, 19–28, to appear.

Bateman, H. (1938), Spheroidal and bipolar coordinates, *Duke Mathematical Journal* **4**, no. 1, 39–50

Bottema, O. (2008), *Topics in Elementary Geometry*, Springer.

Dietz, R., Hoschek, J., Jüttler, B. (1993), An algebraic approach to curves and surfaces on the sphere and on other quadrics, *Computer Aided Geometric Design* **10**, 211–229.

Farin, G. (1999), NURBS. From projective geometry to practical use. AK Peters, Natick MA.

Farin, G. (2002), *Curves and Surfaces for CAGD.* Morgan Kaufmann.

Farin, G. (2006), Rational quadratic circles are parametrized by chord length. *Computer Aided Geometric Design* **23**, 722–724.

Farouki, R., Moon, H.P. (2000), Bipolar and multipolar coordinates, in: *The Mathematics of Surfaces IX* (R. Cippola and R. Martin eds.), Springer, 348–371.

Farouki, R. (2008), Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable, Springer, Berlin.

Lü, W. (2009), Curves with chord length parameterization. *Computer Aided Geometric Design*, **26**, 342–350.

Sabin, M.A., Dodgson, N.A. (2005), A circle-preserving variant of the four-point subdivision scheme. In: Lyche, T., Schumaker, L. (Eds.), *Mathematical Methods for Curves and Surfaces*, Nashboro Press, 275–286.

Sánchez-Reyes, J., Fernández-Jambrina, L. (2008), Curves with rational chord-length parametrization. *Computer Aided Geometric Design* **25**, 205–213.

Teller, S.J., Séquin, C. (1991). Constructing Easily Invertible Bézier Surfaces that Parametrize General Quadrics, Proc. Symposium on Solid Modeling Foundations and CAD/CAM Applications, ACM Press, 303–315.