Decomposing Envelopes of Rational Hypersurfaces

Tino Schulz and Bert Jüttler

Abstract The envelope of a family of real, rational hypersurfaces is defined by an implicit equation in the parameter space. This equation can be decomposed into factors that are mapped to varieties of different dimension. The factorization can be found using solely gcd computations and polynomial divisions. The decomposition is used to derive some general results about envelopes, which also contribute to the analysis of self-intersections.

Key words: envelopes, singularities, self-intersection

1 Introduction

Envelopes of curves and surfaces are a classical topic of differential geometry and kinematics [2, 7]. Due to their importance in various applications, computational techniques for dealing with envelopes have attracted the interest of researchers from several fields. These include robotics (collision detection and avoidance) and gearing theory (design of matching pairs of gear teeth surfaces), geometrical optics (caustics), NC-machining (offset curves for tool path generation) and Computer Aided Geometric Design (sweeps, convolutions, Minkowski sums). See e.g. [1, 5, 6, 8, 9, 10] and the references cited therein.

In this paper, we generalize the approach presented in [11], which is restricted to the curve case, to envelopes of general families of hypersurfaces. More precisely, we will focus on the fact that envelopes are essentially *singularities* of the mapping that

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describes the defining family of hypersurfaces. We will identify and compute the parts of the envelope possessing different dimension, and we derive some general results of envelopes.

2 Envelopes of Rational Hypersurfaces

We recall basic properties of envelopes and show how to identify their parts possessing different dimensions. Consider a rational mapping

$$\mathbf{x}(\mathbf{t}) = \big(x_1(\mathbf{t})/x_0(\mathbf{t}), \ldots, x_n(\mathbf{t})/x_0(\mathbf{t})\big)^\top \tag{1}$$

where the $x_i(\mathbf{t})$ $(i = 0 \ldots n)$ are real, n-variate polynomials. Here

$$\mathbf{t} = (t_1, \ldots, t_n)^\top \in I_1 \times \cdots \times I_n = \mathbf{I} \subset \mathbb{R}^n \tag{2}$$

with closed real intervals I_i $(i = 1 \ldots n)$. Moreover, we assume that $x_0 \neq 0$ for $\mathbf{t} \in \mathbf{I}$, $\gcd(x_0, \ldots, x_n) = 1$ and that the image $\mathbf{x}(\mathbf{I})$ is not completely contained in any hypersurface. Though we are mostly interested in real properties of $\mathbf{x}$, it will be necessary to consider the complex extension of $\mathbf{x} : \mathbb{C}^n \to \mathbb{C}^n$. When not explicitly stated differently we will make use of a complex variable $\mathbf{s} \in \mathbb{C}^n$ and examine $\mathbf{x}(\mathbf{s})$ in the remainder of this paper.

If we pick any index j and use t_j as a time-like parameter $t_j = \tau$, then the mapping $\mathbf{x}$ defines a family of rational hypersurfaces. For each value of τ, the corresponding hypersurface is obtained by varying the remaining $n-1$ parameters t_i $(i \neq j)$.

The envelope of this family of hypersurfaces is defined by the property that it is tangent to almost every member of the family. With respect to the mapping $\mathbf{x}$, we can characterize the envelope as the image of those points where the Jacobian $\mathbf{J}$ is singular, i.e., *envelopes* are essentially *singularities*. Consequently, the envelope is independent of the choice of the index j. A short computation confirms that

$$\det \mathbf{J} = \frac{1}{x_0} \begin{vmatrix} x_0 & 0 & \cdots & 0 \\ x_1 & \frac{\partial_1 x_1}{x_0} - \frac{\partial_1 x_0}{x_0^2} x_1 & \cdots & \frac{\partial_n x_1}{x_0} - \frac{\partial_n x_0}{x_0^2} x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & \frac{\partial_1 x_n}{x_0} - \frac{\partial_1 x_0}{x_0^2} x_n & \cdots & \frac{\partial_n x_n}{x_0} - \frac{\partial_n x_0}{x_0^2} x_n \end{vmatrix} = \frac{1}{x_0^{n+1}} \underbrace{\begin{vmatrix} x_0 & \partial_1 x_0 & \cdots & \partial_n x_0 \\ x_1 & \partial_1 x_1 & \cdots & \partial_n x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & \partial_1 x_n & \cdots & \partial_n x_n \end{vmatrix}}_{= h} \tag{3}$$

where ∂_i denotes the differentiation with respect to the i-th variable. The determinant defining the polynomial h is obtained by adding $(\partial_i x_0)/x_0^2$ times the leftmost column to the i-th one for $i = 1, \ldots, n$, and then factoring out the common denominators.

Since the points $\mathbf{s} \in \mathbb{C}^n$ satisfying $h(\mathbf{s}) = 0$ are mapped to the envelope, we call h the *envelope function*. The zero set of h consists of one or several (possibly complex) hypersurfaces, i.e. surfaces whose dimension is exactly $n-1$.

3 Decomposing Envelopes

The envelope function possesses a factorization in $\mathbb{C}[\mathbf{s}]$ into irreducible and relatively prime polynomials h_j, $j = 1, \ldots, M$ with certain multiplicities. By a suitable ordering we can guarantee that the first N factors ($N \leq M$) do not divide x_0, while the remaining ones do. After eliminating all factors shared with x_0 and reducing the multiplicities of the remaining factors to 1 we obtain the *reduced envelope function*

$$\hat{h}(\mathbf{s}) = \prod_{j=1,\ldots,N} h_j(\mathbf{s}). \tag{4}$$

Instead of using factorization techniques, the reduced envelope function can be found via suitable gcd computations, cf. [3]. Clearly, $\hat{h}$ is squarefree and the gradients satisfy $(\nabla_{\mathbf{s}}\hat{h})(\mathbf{s}) \neq \mathbf{0}$ almost everywhere, where $\nabla_{\mathbf{s}}$ denotes the gradient with respect to $\mathbf{s}$.

The zero sets of the factors h_j are mapped to components possessing different dimensions. In order to identify those factors h_j whose zero sets are mapped to varieties of a certain dimension, we consider the restriction of the differential of $\mathbf{x}$ at a point $\mathbf{s}$ to the tangent spaces of these zero sets. If the rank of this restriction is equal to r for almost all points satisfying $h_j(\mathbf{s}) = 0$, then this algebraic variety is mapped to a variety of dimension r.

The differential of $\mathbf{x}$ at $\mathbf{s}$ is the linear mapping defined by the Jacobian $\mathbf{J}(\mathbf{s})$. We consider the augmented Jacobian $\mathbf{J}^+(\mathbf{s})$ which is obtained by adding the row vector $\left((\nabla_{\mathbf{s}}\hat{h})(\mathbf{s})\right)^\top$ to $\mathbf{J}(\mathbf{s})$. The augmented Jacobian thus has $n+1$ rows and n columns.

The dimension of the kernel of $\mathbf{J}^+$ equals $n-r$, where $r = \operatorname{rk}\mathbf{J}$. For all points $\mathbf{s}_0 \in \mathbb{C}^n$ satisfying $(\nabla_{\mathbf{s}}\hat{h})(\mathbf{s}_0) \neq \mathbf{0}$, the hypersurface $\hat{h}(\mathbf{s}) = \hat{h}(\mathbf{s}_0)$ possesses a well-defined tangent space at $\mathbf{s}_0$ and the kernel of $\mathbf{J}^+(\mathbf{s}_0)$ is contained in it, due to the additional row in the augmented Jacobian. Consequently, the augmented Jacobian $\mathbf{J}^+(\mathbf{s})$ – and hence also the Jacobian $\mathbf{J}(\mathbf{s})$ – maps this tangent space into a space of dimension $r-1$.

Thus, for almost all points satisfying $h_j(\mathbf{s}) = 0$ for a particular index j, the dimension of the image of this hypersurface under $\mathbf{x}$, i.e., of the associated component of the envelope, is equal to $\operatorname{rk}\mathbf{J}(\mathbf{s}) - 1$. This property is inherited by the matrix $\mathbf{V} = (x_0)^2\mathbf{J}^+$, which has polynomial entries. The vanishing of all i-th order minors of $\mathbf{V}$ (the determinants of all its $(i \times i)$-submatrices) is a necessary and sufficient condition for $\operatorname{rk}\mathbf{V} \leq i-1$.

This observation leads us to formulate the following procedure for decomposing $\hat{h}$ into factors whose zero sets are mapped into components of different dimensions:

- Let $g_{n+1} = \hat{h}$. Further, let g_i be the greatest common divisor of $\hat{h}$ and of all i-th order minors of $\mathbf{V}$ $(i = 1 \ldots n)$. Obviously, g_i divides g_{i+1}, and the zero set of g_i is mapped into components of maximum dimension $i-2$.
- Further, let $f_i = g_{i+2}/g_{i+1}$, $(i = 0 \ldots n-1)$. The zero set of f_i is mapped into components of dimension i. This gives the decomposition

$$\hat{h}(\mathbf{s}) = \prod_{i=0}^{n-1} f_i(\mathbf{s}). \tag{5}$$

The polynomial f_{n-1} is called the *proper envelope function*. A factor h_j of $\hat{h}$ is called *proper*, if and only if it is also a factor of f_{n-1}, otherwise it is said to be *improper*.

We summarize these observations in

Theorem 1. *The rational mapping* $\mathbf{x}$ *maps the zero set of the polynomial* f_i *into a component of the envelope which is an algebraic variety of dimension i or a collection of several such varieties.*

Let $\mathscr{D}_i$ be be the image of the zero set of f_i under $\mathbf{x}$ ($i = 0,\ldots,n-1$). The sets $\mathscr{D}_i$ are images of real algebraic hypersurfaces under a real rational mapping and are of complex dimension i. Their real dimension might be lower.

The *proper part* $\mathscr{D}_{n-1}$ of the envelope is of particular interest. There exists a real, squarefree polynomial q such that

$$\mathscr{D}_{n-1} \subseteq V(q) = \{\mathbf{p} \in \mathbb{C}^n : \ q(\mathbf{p}) = 0\} \tag{6}$$

i.e. $q = 0$ is an implicit equation of the envelope. The following example illustrates these facts.

Example 1. Let $n = 3$ and consider

$$\mathbf{x}(s,t,u) = \left(\frac{(s+t)(st+1)(u-1)}{1+s^2}, \frac{4u}{1+u^2}, \frac{s^2(1+t^2)(1-u^2)^2}{(1+s^2)(1+u^2)} \right)^{\top}. \tag{7}$$

Its reduced envelope function is $\hat{h} = (t+I)(t-I)(1+st)(u+1)(u-1)s$,where $I^2 = -1$. A short computation gives

$$f_0 = u-1, \quad f_1 = (t+I)(t-I)(1+st)(u+1) \quad \text{and} \quad f_2 = s. \tag{8}$$

By applying $\mathbf{x}$ to the zero sets of the polynomials f_i we obtain that

- $\mathscr{D}_0$ is the point $(0,2,0)^{\top}$,
- $\mathscr{D}_1$ consists of an ellipse, a line, and two complex conjugate ellipses and
- $\mathscr{D}_2$ is a certain subset of the xy-plane.

Consequently, we get $q(x,y,z) = z$. The numerator of $q \circ \mathbf{x}$ includes all those factors of $\hat{h}$ that are mapped on the proper part of the envelope. Note that f_2 and two additional factors appear in $q \circ \mathbf{x}$ with multiplicity 2. This will be investigated in the next section.

The computation of the exact implicit equation of an envelope is rather expensive in general. Although several methods exist [3], their complexity usually restricts their practical application to planar or low-degree problems. Techniques for *approximate implicitization* are a valuable alternative, see [4, 11].

4 Using the Decomposition

In this section, we will use the factorization (5) to derive several properties of the envelope. The first result generalizes Theorem 1 from [11].

Theorem 2. *Let q be the implicit equation of the proper part of the envelope as defined in* (6)*. There exists a real, n-variate, polynomial $\tilde{\lambda} : \mathbb{C}^n \to \mathbb{C}$, such that*

$$(q \circ \mathbf{x}) \cdot (x_0)^d = \tilde{\lambda} \cdot (f_{n-1})^2, \tag{9}$$

where d is the degree of q.

Proof. Since $\mathscr{D}_{n-1} \subset V(q)$ consists of an $n-1$-dimensional family of points $\mathbf{x}(\mathbf{s})$ fulfilling $f_{n-1}(\mathbf{s}) = 0$, we can conclude that f_{n-1} is a factor of the numerator of $q \circ \mathbf{x}$. Additionally we note that if $f_{n-1}(\mathbf{s}) = 0$, then

$$\nabla_{\mathbf{s}} (q \circ \mathbf{x})(\mathbf{s}) = \mathbf{J}(\mathbf{s})^{\top} \big(\nabla_x q \circ \mathbf{x}\big)(\mathbf{s}) = \mathbf{0}, \tag{10}$$

because $\mathbf{J}(\mathbf{s})$ spans the tangent space of the envelope. This implies that $(f_{n-1})^2$ is a factor of the numerator of $q \circ \mathbf{x}$ since f_{n-1} is squarefree. □

Theorem 2 implies that $(f_{n-1})^2$ is a factor of the composition $q \circ \mathbf{x}$. Now we study the remaining factors of multiplicity 2:

Corollary 1. *If $\tilde{\lambda}$ is not squarefree, then its factors of multiplicity greater than one are also factors of $\hat{h}$.*

Proof. If $\tilde{\lambda}$ is not squarefree then there exist polynomials $\nu, \mu : \mathbb{C}^n \to \mathbb{C}$ such that $\tilde{\lambda} = \nu\mu$, where ν is squarefree and μ has only factors of multiplicity greater than one. For every $\mathbf{s} \in \mathbb{C}^n$ with $\mu(\mathbf{s}) = 0 \neq x_0(\mathbf{s})$ we get that

$$(q \circ \mathbf{x})(\mathbf{s}) = \nu(\mathbf{s})\mu(\mathbf{s})\big(f_{n-1}(\mathbf{s})\big)^2 = 0 \tag{11}$$

which implies

$$\nabla_{\mathbf{s}} (q \circ \mathbf{x})(\mathbf{s}) = \mathbf{J}(\mathbf{s})^{\top} \big(\nabla_x q \circ \mathbf{x}\big)(\mathbf{s}) = \mathbf{0}, \tag{12}$$

since $\nabla_{\mathbf{s}}\mu(\mathbf{s}) = \mathbf{0}$. The rightmost identity of equation (12) can only be fulfilled for a $n-1$-dimensional family of points if $\mathbf{J}$ is singular. Thus every factor of the squarefree representation of μ must also be a factor of $\hat{h}$. □

Consequently, the factors of $\tilde{\lambda}$ with a multiplicity greater than one correspond to those factors of $\hat{h}$ that are "singularly" mapped on the proper part of the envelope. Note that $\tilde{\lambda}$ might contain factors of x_0 which we eliminate by setting

$$\lambda = \tilde{\lambda} / \gcd(\tilde{\lambda}, x_0). \tag{13}$$

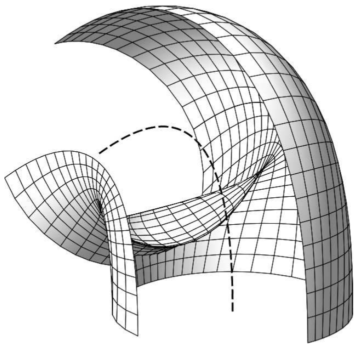

Fig. 1 Example 2: A part of the envelope, which is the offset of a parabola (dashed curve), and its self-intersections.

5 Self-intersections and "Undercuts"

According to Theorem 2, $q \circ \mathbf{x} = 0$ holds also for every point on the zero set of λ. The factors of λ which are not factors of f_{n-1} characterize additional intersections of the family $\mathbf{x}$ with the proper part of the envelope:

Corollary 2. *Let* $\mathbf{s}' \in \mathbb{C}^n$ *such that* $f_{n-1}(\mathbf{s}') \neq 0 \neq x_0(\mathbf{s}')$. *Assume there exists* $\mathbf{s} \in \mathbb{C}^n$ *satisfying* $f_{n-1}(\mathbf{s}) = 0$ *and* $\mathbf{x}(\mathbf{s}) = \mathbf{x}(\mathbf{s}')$. *Then* $\lambda(\mathbf{s}') = 0$.

Proof. We directly obtain $0 = (q \circ \mathbf{x})(\mathbf{s}) = (q \circ \mathbf{x})(\mathbf{s}') = \lambda(\mathbf{s}') \cdot f_{n-1}(\mathbf{s}')$, which implies $\lambda(\mathbf{s}') = 0$. □

Consequently, the $\lambda(\mathbf{s}') = 0$ is a necessary condition for the point $\mathbf{x}(s')$ to be located on the proper part of the envelope, and therefore to create an "undercut". This interesting observation may be used for the trimming of offsets and for eliminating the undercut of envelope surfaces. We explain this by an example.

Example 2. Consider the rational mapping

$$\mathbf{x}(s,t,u) = \frac{1}{x_0(s,t)} \begin{pmatrix} \left(1+s^2\right)\left(u+ut^2+2t\right) \\ 2s(1+t)(1-t) \\ s^2+t^2-u^2-s^2t^2-s^2u^2-t^2u^2-s^2t^2u^2-1 \end{pmatrix}, \tag{14}$$

with $x_0 = \left(1+s^2\right)\left(1+t^2\right)$. It describes as a sphere of radius 1 whose center is moving along a parabola in the xz-plane, where u is the time-like parameter.

The proper envelope function is $f_{n-1} = (1-s^2-t^2-s^2t^2)u + (1+t)s^2$ and the proper part of the envelope is the offset surface of distance 1 of the parabola, see Fig. 1. It is a pipe surface with the implicit equation

$$\begin{aligned} q(x,y,z) = \; & 16(x^2+y^2)^2(x^2+y^2+z^2) - 2z(3x^2-36y^2-20z^2) \\ & +8z(5x^4-4y^4+x^2y^2+4x^2z^2-4y^2z^2) + 28x^2+65y^2+9z^2 \\ & -47x^4-56y^4+16z^4-76x^2y^2-24y^2z^2-40z-25, \end{aligned} \tag{15}$$

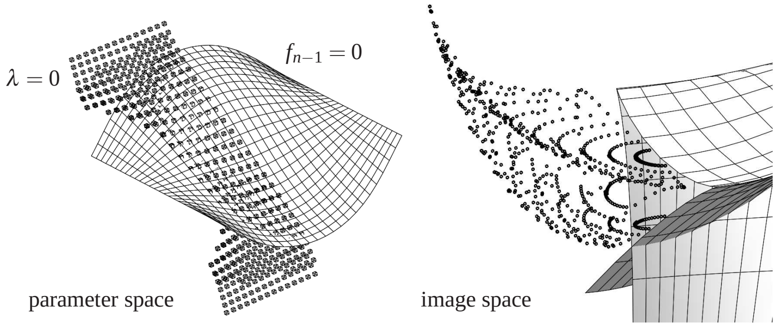

Fig. 2 Example 2: The zero sets of f_{n-1} (shows as surface) and λ (shown as point cloud) in the parameter space (left) are mapped onto the envelope $\mathscr{D}_2$ in the image space (right). In particular, the zero set of λ s mapped to the undercut region.

and it possesses a certain region of self intersection. If one thinks of $\mathbf{x}$ as describing a moving cutting tool which moves along some path, then the part of the envelope that is bounded by its singularities would be cut away. Thus, in situations like in this example, this part is referred to as *undercut*. In several applications (e.g. offset trimming), it is an important task to determine it.

Let $\lambda(s,t,u)$ be defined as in section 4, i.e. take $\tilde{\lambda} = (q \circ \mathbf{x}) \cdot (x_0)^6/(f_{n-1}^2)$ and remove common factors with x_0. It is a rather complicated polynomial of tri-degree $(6,6,4)$ which describes two surfaces that are almost parallel. Figure 2 shows the zero sets of λ and f_{n-1} in parameter space (which are visualized by a point cloud and by parameter lines, respectively) and their images under $\mathbf{x}$. The image of $\lambda = 0$ is the undercut region, and the curves defined by $\lambda = f_{n-1} = 0$ are mapped to the the self-intersection curves of the envelope.

The additional components which are defined by λ also appear in the problem of sorting assembly modes in robot kinematics. In that context they are referred to as *characteristic (hyper-)surfaces*, see [12].

6 Conclusion

We have shown how to decompose the defining equations of an envelope into polynomial factors that are mapped onto varieties of different dimension. The proposed method is algorithmically simple and constructs an explicit decomposition only using gcd computations and polynomial division.

We then deduced some general properties of envelopes, generalizing existing results for curves. In particular, we addressed some aspects which are closely related to the analysis of self-intersections and “undercuts”.

Future work could be devoted to a more detailed investigation on the properties of the factorization described in Theorem 2, to its application in the determination of undercut regions, and to the use of approximate implicitization techniques for envelope surfaces.

Acknowledgments. The first author was supported by the Marie-Curie Network SAGA (FP7, GA no. 214584), and by the Doctoral Program “Computational Mathematics” (W1214) at Johannes Kepler University, Linz. The authors thank the anonymous referees for their useful comments, in particular for their contributions that led to a correct version of Theorem 1.

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