H^2 regularity properties of singular parameterizations in isogeometric analysis

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Abstract

Isogeometric analysis (IGA) is a numerical simulation method which is directly based on the NURBS-based representation of CAD models. It exploits the tensor-product structure of 2- or 3-dimensional NURBS objects to parameterize the physical domain. Hence the physical domain is parameterized with respect to a rectangle or to a cube. Consequently, singularly parameterized NURBS surfaces and NURBS volumes are needed in order to represent non-quadrangular or non-hexahedral domains without splitting, thereby producing a very compact and convenient representation.

The Galerkin projection introduces finite-dimensional spaces of test functions in the weak formulation of partial differential equations. In particular, the test functions used in isogeometric analysis are obtained by composing the inverse of the domain parameterization with the NURBS basis functions. In the case of singular parameterizations, however, some of the resulting test functions do not necessarily fulfill the required regularity properties. Consequently, numerical methods for the solution of partial differential equations can not be applied properly.

We discuss the regularity properties of the test functions. For one- and two-dimensional domains we consider several important classes of singularities of NURBS parameterizations. For specific cases we derive additional conditions which guarantee the regularity of the test functions. In addition we present a modification scheme for the discretized function space in case of insufficient regularity. It is also shown how these results can be applied for computational domains in higher dimensions that can be parameterized via sweeping.

1. Introduction

The product development process in engineering often involves two major phases. In the first phase, a geometric model of the product is constructed. This is based on tools from Computer Aided Design (CAD), where the geometry is represented by B-splines or by non-uniform rational B-splines (NURBS). The second phase deals with the numerical simulation of processes such as heat transfer, the computation of pressure or stress distributions or the analysis of fluid flow. This simulation phase is usually performed numerically by means of the Finite Element Method (FEM).

The classical finite element method works on meshes, consisting of geometric primitives like triangles, quadrilaterals, tetrahedra or hexahedra. Therefore one has to derive such a computational mesh from the NURBS representation of the geometry. The isogeometric method, introduced by Hughes et al. [1], does not need this transformation step, since it directly uses the NURBS representation to build up a function space for numerical simulations.

Various applications of isogeometric analysis (IGA) have been studied so far, for instance problems in fluid dynamics [2–4], in shape optimization [5–7] and modeling the deformation of solids [8–10]. Contributions to the theoretical background of the isogeometric method treat the numerical analysis concerning consistency and stability of the method [11–14]. Usually, the case of singularly parameterized domains is not covered.

Nevertheless, singular parameterizations are of great use for the modeling of physical domains and have to be treated separately. Singularities in the parameterization can be caused by distortions of regular parameterizations or by intrinsic properties of the geometry, which cannot be avoided in many situations. Since higher dimensional NURBS possess a tensor-product structure they can only describe quadrangular or hexahedral domains directly without the use of singularities. If a single-patch parameterization is used to directly represent a non-quadrangular or non-hexahedral domain like a circle or a sphere, then singularities are necessary [15–17]. A different approach to represent general domains uses the concept of weighted extended B-splines (web-splines) introduced in [18]. In that case a spline space is defined on a larger domain which is then properly trimmed to the boundary of the desired domain. Customly trimmed surfaces and volumes are also widely used to parameterize domains without using singularities. Since stability issues might occur for function spaces on trimmed domains, we do not go into the details of trimming techniques.

We will consider isogeometric analysis as a solution method for partial differential equations. In this context we focus on equations that lead to the underlying function spaces H^1 and H^2. The space H^1 is the basic function space when considering variational formulations of second order partial differential equations. The function space H^2 is needed when considering certain higher order equations, such as the biharmonic equation, which may occur for applications in linear elasticity theory or in Stokes flow (see e.g. [19] for an application in isogeometric analysis).

In this work we do not consider NURBS but restrict ourselves to B-splines. The results that are obtained for B-splines can be generalized to NURBS parameterizations fulfilling certain conditions as defined in Section 3.3 of [14]. The focus lies on the applicability of the numerical methods in the case of singularly parameterized domains. We concentrate on the regularity properties of isogeometric test functions. An isogeometric test function is the composition of a B-spline with the inverse of the domain parameterization. Since the parameterization is assumed to be singular in some points the test function may not be well defined. Hence it may not be sufficiently regular. For various cases some of the test functions are not in the desired function space, in our case H^1 or H^2. The H^1-case has been analyzed in [14]. In the present paper we concentrate on H^2 regularity. While many of the techniques used in the previous paper are still applicable, the theory and the results become much more complex.

There exist results concerning isoparametric elements with singularities in the context of finite element methods. In [20, 21] singular isoparametric finite elements are used to approximate singularities in the solution. The results for such finite elements could be generalized to B-spline parameterizations, but the problems and results presented there differ from the problems considered in this paper. There also exist some results for degenerated finite elements (e.g. [22, 23]) where bounds for interpolation errors are stated. The results presented there are related to this paper but cover only bilinear elements and cannot be generalized directly to higher degree patches.

The next section gives a short introduction to isogeometric analysis. In Section 3 we develop the theory for $1D$ domains and in Section 4 for $2D$ domains. Section 5 presents a framework to analyze regularity properties for more general domains using the concept of structural equivalence. Finally we conclude the paper with a short summary and an outlook to topics that may be of interest for future research.

2. Preliminaries

In this section we will present the basics of isogeometric analysis. We will adopt the same notation as in [14]; some of the definitions will be recalled now.

2.1. Variational formulation and H^i-norms

Let $\Omega \subseteq \mathbb{R}^d$ be a d-dimensional domain and let $\mathcal{V}_g(\Omega), \mathcal{V}_0(\Omega) \subseteq \mathcal{V}(\Omega)$ be certain subsets (defined by imposing suitable boundary conditions) of a Hilbert space $\mathcal{V}(\Omega)$. Given a bilinear form $a(\cdot,\cdot) : \mathcal{V}_g \times \mathcal{V}_0 \to \mathbb{R}$ and a linear functional $\langle F, \cdot\rangle : \mathcal{V}_0 \to \mathbb{R}$ we consider a variational formulation of a partial differential equation:

$$\text{Find } u \in \mathcal{V}_g(\Omega) \text{ such that } a(u,v) = \langle F, v\rangle \qquad \forall\ v \in \mathcal{V}_0(\Omega).$$

We refer to [24] for a more detailed analysis and description of the problem. We will restrict ourselves to $\mathcal{V}(\Omega) = H^1(\Omega)$ or $\mathcal{V}(\Omega) = H^2(\Omega)$ as the underlying Hilbert space. The function spaces $H^1(\Omega)$ and $H^2(\Omega)$ are defined by

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial \xi_k} \in L^2(\Omega)\ \forall\, 1 \le k \le d \right\}$$

and

$$H^2(\Omega) = \left\{ v \in H^1(\Omega) : \frac{\partial^2 v}{\partial \xi_k \partial \xi_l} \in L^2(\Omega)\ \forall\, 1 \le l,k \le d \right\},$$

where the derivatives have to be interpreted in a weak sense. With the use of the H^1- and H^2-seminorms

$$|v|_{H^1} = \left(\sum_{k=1}^{d} \left\| \frac{\partial v}{\partial \xi_k} \right\|_{L^2}^2 \right)^{1/2} \quad \text{and} \quad |v|_{H^2} = \left(\sum_{k,l=1}^{d} \left\| \frac{\partial^2 v}{\partial \xi_k \partial \xi_l} \right\|_{L^2}^2 \right)^{1/2}$$

the Hilbert space norms in H^1 and H^2 are defined via

$$||v||^2_{H^1} = ||v||^2_{L^2} + |v|^2_{H^1} \quad \text{and} \quad ||v||^2_{H^2} = ||v||^2_{L^2} + |v|^2_{H^1} + |v|^2_{H^2} \,.$$

It is obvious that these norms are well-defined if and only if the function is in H^1 or H^2, respectively.

2.2. *Galerkin discretization in isogeometric analysis*

The isogeometric method is an approach to discretize partial differential equations on non-trivial geometries derived from CAD systems. It is based on Galerkin's principle, which can be interpreted in the following way. Having a finite-dimensional function space $\mathcal{V}_h \subseteq \mathcal{V}$ the spaces $\mathcal{V}_{g,h} = \mathcal{V}_g \cap \mathcal{V}_h$ and $\mathcal{V}_{0,h} = \mathcal{V}_0 \cap \mathcal{V}_h$ are set up to solve the following discretized problem:

$$\text{Find} \quad u_h \in \mathcal{V}_{g,h}(\Omega) \quad \text{such that} \quad a(u_h, v_h) = \langle F, v_h \rangle \qquad \forall v_h \in \mathcal{V}_{0,h}(\Omega)\,.$$

The choice of the discrete subspace $\mathcal{V}_h$ (or its basis functions) is called a Galerkin discretization. In our setting the basis functions spanning $\mathcal{V}_h$ are constructed from B-splines, which are piecewise polynomials, defined over some parameter space $\mathbf{B} \subseteq \mathbb{R}^d$. For a precise and detailed theoretical background on B-splines and NURBS in computer aided geometric design we refer the reader to [25–27].

Let $B_{i,p}$ be the ith B-spline of degree $p \in \mathbb{N}$ with the knot vector $\Theta = (\theta_0, \ldots, \theta_{m-1})$. The parameter space is set to be $\mathrm{B} = \left]\theta_p, \theta_{m-p-1}\right[$, which covers the support of each B-spline, except for the boundary intervals $\left[\theta_0, \theta_p\right]$ and $\left[\theta_{m-p-1}, \theta_{m-1}\right]$.

In order to extend the concept of B-splines to two dimensions one can introduce bivariate tensor product B-splines. Consequently, a degree and a knot vector is set for each direction. We consider a degree $\mathbf{p} = (p_1, p_2)$, a knot vector $\mathbf{\Theta} = \left(\Theta^{(1)}, \Theta^{(2)}\right)$, with $\Theta^{(1)} \in \mathbb{R}^{m_1}$ and $\Theta^{(2)} \in \mathbb{R}^{m_2}$, and set $(n_1, n_2) = \mathbf{n} = \mathbf{m} - \mathbf{p} - \mathbf{1}$. Using the notation $\mathbf{i} = (i, j)$ and $\mathbf{x} = (x, y)^T$, then $\mathrm{B}_{\mathbf{i,p}}$ is the $\mathbf{i}$th bivariate B-spline of degree $\mathbf{p}$ and knot vector $\mathbf{\Theta}$ for $\mathbf{0} \leq \mathbf{i} \leq \mathbf{n} - \mathbf{1}$. The parameter space $\mathbf{B}$ is defined by

$$\mathbf{B} = \left]\theta^{(1)}_{p_1}, \theta^{(1)}_{m_1-p_1-1}\right[\times \left]\theta^{(2)}_{p_1}, \theta^{(2)}_{m_2-p_2-1}\right[\,.$$

In order to compactly describe our results, we will use a notation which is independent of the dimension d of the physical space Ω, but follows the notational standards for the multivariate case.

Without loss of generality we choose the parameter domain to be the d-dimensional open unit box $\mathbf{B} =]0,1[^d$. We set the index space $\mathbb{I}$ to

$$\mathbb{I} = \left\{\mathbf{i} \in \mathbb{N}^d : \mathbf{0} \leq \mathbf{i} \leq \mathbf{n} - \mathbf{1}\right\}.$$

The parameterization $\mathbf{G}$ of Ω is defined by

$$\mathbf{G} : \mathbf{B} \;\rightarrow\; \mathbb{R}^d : \quad \mathbf{x} \;\mapsto\; \sum_{\mathbf{i} \in \mathbb{I}} \mathbf{P_i} \phi_{\mathbf{i}}(\mathbf{x}) \,,$$

with B-spline basis functions $\phi_{\mathbf{i}} = \mathrm{B}_{\mathbf{i,p}} : \mathbf{B} \rightarrow \mathbb{R}$ and control points $\mathbf{P_i} \in \mathbb{R}^d$ for each $\mathbf{i} \in \mathbb{I}$. The physical domain Ω is represented as the image of $\mathbf{B}$ under $\mathbf{G}$, i.e. $\mathbf{G}(\mathbf{B}) = \Omega$. We consider basis functions

$$\phi_{\mathbf{i}} : \mathbf{B} \;\rightarrow\; \mathbb{R} : \quad \mathbf{x} \;\mapsto\; \mathrm{B}_{\mathbf{i,p}}(\mathbf{x})$$

on the parameter space. In case of a bijective and continuously differentiable parameterization $\mathbf{G}$ (with C^1-inverse) the test functions, i.e. the basis functions of the function space $\mathcal{V}_h \subset \{v : \Omega \rightarrow \mathbb{R}\}$, are defined by

$$\psi_{\mathbf{i}} : \Omega \;\rightarrow\; \mathbb{R} : \quad \boldsymbol{\xi} \;\mapsto\; \phi_{\mathbf{i}} \circ \mathbf{G}^{-1}(\boldsymbol{\xi})$$

on the physical domain. Figure 1 illustrates the definition of the functions $\mathbf{G}$, $\phi_{\mathbf{i}}$ and $\psi_{\mathbf{i}}$.

Now we can define the isogeometric space of test functions on the physical domain by

$$\mathcal{V}_h = \mathrm{span}_{\mathbf{i} \in \mathbb{I}} \left\{\mathrm{B}_{\mathbf{i,p}} \circ \mathbf{G}^{-1}\right\}.$$

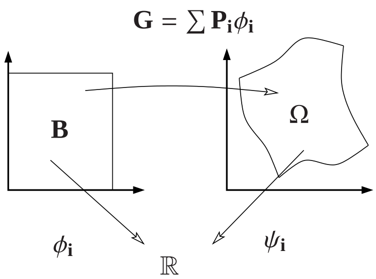

Figure 1: Two-dimensional parameterization $\mathbf{G}$ with parameter domain $\mathbf{B}$, physical domain Ω and basis functions $\phi_\mathbf{i}$ and $\psi_\mathbf{i}$

In order to obtain well–defined functions on the physical domain the parameterization $\mathbf{G}$ has to be invertible in the open box $\mathbf{B}$. Nonetheless it may be singular in some points $x_0 \in \overline{\mathbf{B}}$. We assume that the parameterization $\mathbf{G}$ is bijective in the interior of the parameter space. In practical applications it might happen that overlaps occur in the geometry mapping, i.e. the parameterization is not bijective. It is not clear how to define proper function spaces on overlapping domains. Considering this kind of singularities would exceed the scope of this paper.

We analyze the test functions from isogeometric analysis in the presence of singularities in the parameterization. It might happen that some of the test functions $\psi_\mathbf{i}$ do not fulfill the required regularity conditions. In many applications conditions like $\psi_\mathbf{i} \in H^1$ or $\psi_\mathbf{i} \in H^2$ are needed. Therefore we restrict ourselves to the study of the H^1- and H^2-norm integrals.

2.3. Evaluation of H^i seminorms ($i = 1, 2$)

Our first aim is to find convenient representations for the integrands in order to bound the integrals. In the case of a regularly parameterized domain all integrals will be bounded as long as the differentiability of the spline space is sufficiently high. This is not generally true if singularities occur. First we provide representations for the H^1- and H^2-norm integrals. Hence our aim is to take a closer look at the squares of the L^2-norm

$$||\psi_\mathbf{i}||^2_{L^2(\Omega)} = \int_\Omega \psi_\mathbf{i}(\boldsymbol{\xi})^2 \, \mathrm{d}\boldsymbol{\xi}, \tag{1}$$

the H^1-seminorm

$$|\psi_\mathbf{i}|^2_{H^1(\Omega)} = \int_\Omega \sum_{n=1}^{d} \left(\frac{\partial \psi_\mathbf{i}}{\partial \xi_n}(\boldsymbol{\xi}) \right)^2 \, \mathrm{d}\boldsymbol{\xi} \tag{2}$$

and the H^2-seminorm

$$|\psi_\mathbf{i}|^2_{H^2(\Omega)} = \int_\Omega \sum_{m,n=1}^{d} \left(\frac{\partial^2 \psi_\mathbf{i}}{\partial \xi_n \partial \xi_m}(\boldsymbol{\xi}) \right)^2 \, \mathrm{d}\boldsymbol{\xi} \tag{3}$$

of the test function $\psi_\mathbf{i}$. Let $J = \det \nabla \mathbf{G}$ be the determinant of the Jacobian of $\mathbf{G}$. Since the parameterization is bijective, the Jacobian determinant J is bounded from above by some constant $\overline{J}$ and from below by 0. A transformation of the integral (1) to the parameter space leads to

$$||\psi_\mathbf{i}||^2_{L^2(\Omega)} \quad = \quad \int_\Omega \psi_\mathbf{i}(\boldsymbol{\xi})^2 \, \mathrm{d}\boldsymbol{\xi} \quad = \quad \int_\mathbf{B} \phi_\mathbf{i}(\mathbf{x})^2 \, J(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

which is bounded in any case. Therefore all test functions are in $L^2(\Omega)$, even in the case of a singularly parameterized domain.

The square of the H^1-seminorm (2) can be transformed to a representation on the parameter domain, as described in [14].

Lemma 2.1 (see [14]) *For $\psi_{\mathbf{i}} = \phi_{\mathbf{i}} \circ \mathbf{G}^{-1}$ we have*

$$|\psi_{\mathbf{i}}|^2_{H^1(\Omega)} = \int_{\mathbf{B}} \|\mathrm{Cof}\nabla\mathbf{G}\nabla\phi_{\mathbf{i}}\|^2 \frac{1}{J}\,\mathrm{d}\mathbf{x},$$

where $\mathrm{Cof}\nabla\mathbf{G}$ *is the matrix of cofactors of* $\nabla\mathbf{G}$.

The essential term of the H^2-norm is the integral (3). We obtain the following result.

Lemma 2.2 *For $\psi_{\mathbf{i}} = \phi_{\mathbf{i}} \circ \mathbf{G}^{-1}$ we have*

$$|\psi_{\mathbf{i}}|^2_{H^2(\Omega)} = \int_{\mathbf{B}} \sum_{m,n=1}^{d} (N_{m,n})^2 \frac{1}{J^5}\,\mathrm{d}\mathbf{x}, \quad \textit{where} \quad N_{m,n} = \sum_{i,j=1}^{d} C_{i,m}C_{j,n}\left(J\frac{\partial^2\phi_{\mathbf{i}}}{\partial x_j \partial x_i} - \sum_{k,l=1}^{d} C_{l,k}\frac{\partial\phi_{\mathbf{i}}}{\partial x_k}\frac{\partial^2 \mathbf{G}_l}{\partial x_j \partial x_i}\right).$$

The matrix C *is the matrix of cofactors of* $\nabla\mathbf{G}$, *i.e.*

$$\left(C_{i,j}\right)_{i,j=1}^{d} = \mathrm{Cof}\nabla\mathbf{G},$$

and J is the Jacobian determinant.

Proof. The proof of this statement is postponed to Appendix 6. □

Note that Lemmas 2.1 and 2.2 are valid for any choice of ϕ and $\mathbf{G}$ fulfilling certain smoothness conditions. The functions ϕ, $\mathbf{G}$ and the inverse of $\mathbf{G}$ need to be twice continuously differentiable in the interior of the parameter domain $\mathbf{B}$ and of the physical domain Ω, respectively.

Until now all the results are valid for general domains since we did not specifically consider a singularly parameterized domain. In the next two sections we analyze the behavior of the integrands in the presence of singularities for one- and two-dimensional domains.

3. Singular parameterizations of a line

In this section we consider a one-dimensional physical domain Ω. For this we prove regularity results and introduce a modification framework for the IGA function spaces.

3.1. Regularity analysis

We analyze the H^1- and H^2-seminorms of the test functions $\psi_{\mathbf{i}}$. The H^1-seminorm integral (2) simplifies to

$$|\psi_i|^2_{H^1(\Omega)} = \int_0^1 \frac{\left(\phi_i'(x)\right)^2}{G'(x)}\,\mathrm{d}x.$$

The following theorem recalls earlier results for a special class of singular parameterizations.

Theorem 3.1 (see [14]) *Let $\alpha \in \mathbb{Z}^+$, with $2 \le \alpha \le p$. If the parameterization G is regular for $x > 0$ and the control points satisfy*

- $P_i = 0$, *for* $0 \le i \le \alpha - 1$, *and*
- $P_\alpha \neq 0$,

then

- $\psi_k \notin H^1(\Omega)$ *for* $0 \le k \le \left\lfloor \frac{\alpha}{2} \right\rfloor$ *and*
- $\psi_k \in H^1(\Omega)$ *for* $k > \left\lfloor \frac{\alpha}{2} \right\rfloor$.

Thus, if a singularity occurs at the boundary of the domain due to coinciding control points then approximately half of the corresponding test functions are not in H^1. A more drastic result can be shown for the H^2-case. The general representation of the H^2-seminorm integral (3) simplifies to

$$|\psi_i|^2_{H^2(\Omega)} = \int_0^1 \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial G}{\partial x} - \frac{\partial \phi_i}{\partial x} \frac{\partial^2 G}{\partial x^2} \right)^2 \left(\frac{\partial G}{\partial x} \right)^{-5} \mathrm{d}x. \tag{4}$$

Using this representation, we can prove the following.

Theorem 3.2 *Consider again the situation of Theorem 3.1. If $\alpha < p$ then*

- $\psi_k \notin H^2(\Omega)$ *for* $0 \leq k \leq \min\left(\left\lfloor \frac{1+3\alpha}{2} \right\rfloor, p\right)$ *and*
- $\psi_k \in H^2(\Omega)$ *for* $k > \min\left(\left\lfloor \frac{1+3\alpha}{2} \right\rfloor, p\right)$.

If $\alpha = p$ then

- $\psi_k \notin H^2(\Omega)$ *for* $0 \leq k \leq p-1$ *and*
- $\psi_k \in H^2(\Omega)$ *for* $k \geq p$.

Proof. We first go through the details for the case $\alpha < p$. It is obvious that $\psi_k \in H^2(\Omega)$ for $k > p$. It follows from Theorem 3.1 that $\psi_k \notin H^2(\Omega)$ for $k \leq \left\lfloor \frac{\alpha}{2} \right\rfloor$. It remains to be shown is that the H^2-seminorm of ψ_k does not exist for $\left\lfloor \frac{\alpha}{2} \right\rfloor < k \leq \left\lfloor \frac{1+3\alpha}{2} \right\rfloor$ and that it is bounded for $\left\lfloor \frac{1+3\alpha}{2} \right\rfloor < k \leq p$. The bounds for the H^2-seminorm follow the same scheme as in the proof of Theorem 3.1, which can be found in [14]. To prove the existence or non-existence of the seminorm is technical but follows directly from the representation of the H^2-seminorm (4), i.e.

$$|\psi_k|^2_{H^2(\Omega)} = \int_0^1 \frac{\mathrm{N}_k^2}{J^5} \,\mathrm{d}x,$$

and of the asymptotic behavior of numerator N_k^2 and denominator J^5 in the neighborhood of the singular point $x_0 = 0$. It follows directly from the asymptotic behavior of G and ϕ_k that there exist constants C and C_k with $J \sim C x^{\alpha-1}$ and $N_k \sim C_k x^{\alpha+k-3}$. Hence the integral is bounded if and only if $2(\alpha + k - 3) \geq 5(\alpha - 1)$, which is equivalent to the statement of the theorem. Note that $N_k \sim C_k x^{\alpha+k-3}$ is not true for $\alpha = p$. The case $\alpha = p$ can be proved similarly so we do not discuss it in detail here. □

Unlike Theorem 3.1, Theorem 3.2 states that *not only test functions corresponding to collapsing control points but also functions corresponding to adjacent control points are not sufficiently regular*. This is of great importance since it has to be taken into account for all practical implementations.

3.2. Modified test functions

We identified situations where some test functions do not fulfill the necessary regularity conditions. Therefore, modification of the function space $\mathcal{V}_h$ is necessary. The following theorems state that linear combinations of the test functions can be used to build function spaces which fulfill the regularity conditions. In the case of H^1 as the underlying function space, the following result can be achieved.

Theorem 3.3 *(see [14]) Consider again the assumptions of Theorem 3.1. Let $A_1 = \left\lfloor \frac{\alpha}{2} \right\rfloor$ and define*

$$\phi_{A_1,1}(x) = \sum_{i=0}^{A_1} \phi_i(x).$$

Let

$$\mathcal{V}_{h,1} = \mathrm{span}_{A_1 \leq i \leq n-1} \{\psi_{i,1}\}$$

with

$$\psi_{A_1,1}(\xi) = \phi_{A_1,1}\left(G^{-1}(\xi)\right)$$
$$\psi_{i,1}(\xi) = \phi_i\left(G^{-1}(\xi)\right) \qquad \text{for } A_1+1 \le i \le n-1.$$

The modified function space fulfills $\mathcal{V}_{h,1} \subseteq \mathcal{V}_h \cap H^1(\Omega)$. The function space $\mathcal{V}_{h,1}$ contains all linear functions.

If we consider H^2-norms, then we will have to sacrifice more degrees of freedom than in the H^1-case. However, two test functions fulfilling the regularity conditions can be reconstructed. This approach is presented in the following theorem.

Theorem 3.4 *Let all assumptions of Theorem 3.1 be valid, let $A_2 = \min\left(\left\lfloor \frac{1+3\alpha}{2} \right\rfloor, p\right)$ and define*

$$\phi_{A_2-1,2}(x) = \sum_{i=0}^{A_2}\left(1-\frac{P_i}{P_{\max}}\right)\phi_i(x)$$

and

$$\phi_{A_2,2}(x) = \sum_{i=0}^{A_2}\frac{P_i}{P_{\max}}\phi_i(x),$$

where $P_{\max} = \max_{0\le i\le A_2}\{P_i\}$. Set

$$\mathcal{V}_{h,2} = \operatorname{span}_{A_2-1\le i\le n-1}\{\psi_{i,2}(\xi)\}$$

with

$$\psi_{A_2-1,2}(\xi) = \phi_{A_2-1,2}\left(G^{-1}(\xi)\right)$$
$$\psi_{A_2,2}(\xi) = \phi_{A_2,2}\left(G^{-1}(\xi)\right)$$
$$\psi_{i,2}(\xi) = \phi_i\left(G^{-1}(\xi)\right) \qquad \text{for } A_2+1 \le i \le n-1.$$

The modified function space fulfills $\mathcal{V}_{h,2} \subseteq \mathcal{V}_h \cap H^2(\Omega)$. The function space $\mathcal{V}_{h,2}$ contains all linear functions.

Proof. The proof of this theorem consists of two parts. First one has to show that $\psi_{i,2}(\xi) \in H^2(\Omega)$ for all $A_2-1 \le i \le n-1$. This follows directly from Theorem 3.2 for $i \ge A_2+1$. Since

$$\phi_{A_2,2}(x) = \frac{1}{P_{\max}}\left(G(x) - \sum_{i>A_2} P_i\phi_i(x)\right)$$

we have

$$\psi_{A_2,2}(\xi) = \frac{1}{P_{\max}}\left(\xi - \sum_{i>A_2} P_i\psi_i(\xi)\right),$$

which is in $H^2(\Omega)$. Similarly,

$$\psi_{A_2-1,2} = 1 - \psi_{A_2,2} - \sum_{i>A_2}\psi_i$$

fulfills $\psi_{A_2-1,2} \in H^2(\Omega)$. Finally we show that $\mathcal{V}_{h,2}$ contains all linear functions. We have

$$P_{\max}\psi_{A_2,2}(\xi) + \sum_{i>A_2} P_i\psi_i(\xi) = \xi,$$

hence $\xi \in \mathcal{V}_{h,2}$. Obviously $1 \in \mathcal{V}_{h,2}$, which completes the proof. $\square$

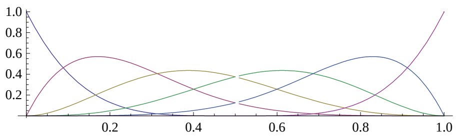

Figure 2: Basis functions ϕ_i on B

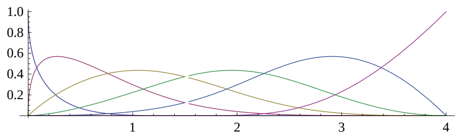

Figure 3: Test functions ψ_i on Ω

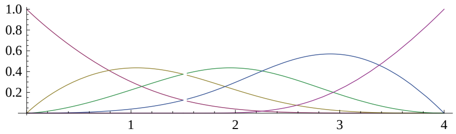

Figure 4: Basis of the function space $\mathcal{V}_{h,1} \subseteq H^1(\Omega)$

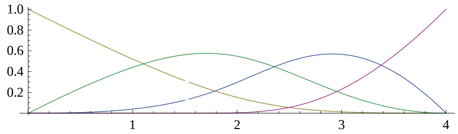

Figure 5: Basis of the function space $\mathcal{V}_{h,2} \subseteq H^2(\Omega)$

Both theorems state that we can modify the function space in order to get the desired regularity. In both cases, however, we reduce the available degrees of freedom, which might lead to worse approximation properties.

Finally we present an example of a singular parameterization.

Example 3.5 Let $p = 4$ be the degree and let $\Theta = \left(0, 0, 0, 0, 0, \frac{1}{2}, 1, 1, 1, 1, 1\right)$ be the knot vector of the B-spline parameterization G. The control points fulfill

$$P_0 = P_1 = 0, \quad P_2 = 1, \quad P_3 = 2, \quad P_4 = 3 \quad \textit{and} \quad P_5 = 4.$$

Figure 2 shows the basis functions ϕ_i on $\mathrm{B} = \,]0, 1[$ and Figure 3 shows the test functions ψ_i on Ω. The next two figures show the basis functions of the modified function spaces. Figure 4 shows the basis of the function space $\mathcal{V}_{h,1}$ as presented in Theorem 3.3. Figure 5 shows the basis of $\mathcal{V}_{h,2}$ as presented in Theorem 3.4. It can be seen that the number of basis functions decreases if higher regularity is needed.

4. Singular parameterizations of planar domains

Until now we only considered one-dimensional domains. Similar results for two-dimensional domains will be presented in this section.

4.1. Regularity analysis

We consider the integrals

$$|\psi_{\mathbf{i}}|^2_{H^1(\Omega)} = \int_\Omega \sum_{n=1}^{2} \left(\frac{\partial \psi_{\mathbf{i}}}{\partial \xi_n}\right)^2 \mathrm{d}\boldsymbol{\xi} \quad \text{and} \quad |\psi_{\mathbf{i}}|^2_{H^2(\Omega)} = \int_\Omega \sum_{m,n=1}^{2} \left(\frac{\partial^2 \psi_{\mathbf{i}}}{\partial \xi_n \partial \xi_m}\right)^2 \mathrm{d}\boldsymbol{\xi},$$

where $\Omega = \mathbf{G}(\mathbf{B})$ with $\mathbf{B} = \,]0, 1[^2$. In order to simplify the representation of the integrals we introduce $F_{\mathbf{i}}$ as the parameterization of the graph of $\psi_{\mathbf{i}}$, i.e.

$$F_{\mathbf{i}}(\mathbf{x}) = \left(\mathbf{G}_1(\mathbf{x}), \mathbf{G}_2(\mathbf{x}), \phi_{\mathbf{i}}(\mathbf{x})\right)^T.$$

We denote partial derivatives of the surface $F_{\mathbf{i}}$ with superscript indices, i.e.

$$F_{\mathbf{i}}^{(k)}(\mathbf{x}) = \frac{\partial F_{\mathbf{i}}}{\partial x_k}(\mathbf{x})$$

and

$$F_{\mathbf{i}}^{(k,l)}(\mathbf{x}) = \frac{\partial^2 F_{\mathbf{i}}}{\partial x_k \partial x_l}(\mathbf{x})$$

at any point $\mathbf{x} \in \mathbf{B}$.

In Lemma 4.1 we rewrite the expansion of the square of the H^1-seminorm of $\psi_{\mathbf{i}}$.

Lemma 4.1 *Considering the square of the H^1-seminorm of $\psi_{\mathbf{i}}$, i.e.*

$$|\psi_{\mathbf{i}}|^2_{H^1(\Omega)} = \int_\Omega \sum_{n=1}^{2} \left(\frac{\partial \psi_{\mathbf{i}}}{\partial \xi_n}\right)^2 \, \mathrm{d}\boldsymbol{\xi},$$

we have

$$|\psi_{\mathbf{i}}|^2_{H^1(\Omega)} = \int_{\mathbf{B}} \left|F^{(1)}_{\mathbf{i}} \times F^{(2)}_{\mathbf{i}}\right|^2 \frac{1}{J} \, \mathrm{d}\mathbf{x} - \int_{\mathbf{B}} J \, \mathrm{d}\mathbf{x}, \tag{5}$$

where $J = \det \nabla \mathbf{G}$. Hence $|\psi_{\mathbf{i}}|^2_{H^1(\Omega)}$ exists if and only if

$$\int_{\mathbf{B}} \left|F^{(1)}_{\mathbf{i}} \times F^{(2)}_{\mathbf{i}}\right|^2 \frac{1}{J} \, \mathrm{d}\mathbf{x} < \infty.$$

The latter is an integral of a rational function.

Proof. The statement can be shown using elementary calculus. □

Note that the numerator $\left|F^{(1)}_{\mathbf{i}} \times F^{(2)}_{\mathbf{i}}\right|^2$ of the fraction is the determinant of the first fundamental form of the parameterized surface $F_{\mathbf{i}}(\mathbf{B})$.

An approach similar to the H^1-seminorm expansion can be applied to the H^2-seminorm of the function $\psi_{\mathbf{i}}$. First we define the tensor $\mathcal{B} = (\mathcal{B}_{k,l})^2_{k,l=1}$ via

$$\mathcal{B}_{k,l} = F^{(k,l)}_{\mathbf{i}} . \left(F^{(1)}_{\mathbf{i}} \times F^{(2)}_{\mathbf{i}}\right). \tag{6}$$

Lemma 4.2 presents a representation of the H^2-seminorm integral.

Lemma 4.2 *Considering the square of the H^2-seminorm of $\psi_{\mathbf{i}}$, i.e.*

$$|\psi_{\mathbf{i}}|^2_{H^2(\Omega)} = \int_\Omega \sum_{m,n=1}^{2} \left(\frac{\partial^2 \psi_{\mathbf{i}}}{\partial \xi_n \partial \xi_m}\right)^2 \, \mathrm{d}\boldsymbol{\xi},$$

we have

$$|\psi_{\mathbf{i}}|^2_{H^2(\Omega)} = \int_{\mathbf{B}} \left\|\mathrm{Cof}\,(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}\,(\nabla \mathbf{G})^T\right\|^2_F \frac{1}{J^5} \, \mathrm{d}\mathbf{x}, \tag{7}$$

where $J = \det \nabla \mathbf{G}$ and $\|\cdot\|_F$ is the Frobenius norm.

Proof. The statement can be shown using elementary calculus. □

Note that the tensor $\mathcal{B}$ is a multiple of the second fundamental form of the surface $F_{\mathbf{i}}(\mathbf{B})$, with the scalar factor $\left|F^{(1)}_{\mathbf{i}} \times F^{(2)}_{\mathbf{i}}\right|$.

Having a representation of the H^1- and H^2-seminorm as integrals of rational functions at hand, we conclude regularity results for instances of B-spline parameterizations. We cannot obtain regularity results for general parameterizations. Instead, we consider certain classes of singular parameterizations and prove the boundedness or unboundedness of the seminorm integrals.

We consider two special cases of B-spline patches. The first case covers patches, where one edge in the parameter domain degenerates to a single point in the physical domain. The second case examines parameterizations, where two adjacent edges in the parameter domain have a common tangent direction at the corner point in the physical domain.

- **Case I: collapsing edge.** Let Ω be a B-spline patch of degree (p_1, p_2). The representation consists of $n_1.n_2$ tensor-product basis functions. The *index set of degeneration* $\mathbb{D} \subseteq \mathbb{I}$ fulfills

  $$\mathbb{D} = \{(i_1, i_2) \in \mathbb{I} : i_1 = 0\}$$

  and the control points fulfill $\mathbf{P}_{\mathbf{i}} = \mathbf{O}$ for $\mathbf{i} \in \mathbb{D}$ and $\mathbf{P}_{\mathbf{i}} \neq \mathbf{O}$ for $\mathbf{i} \in \mathbb{I}\backslash\mathbb{D}$. The parameterization $\mathbf{G}$ is singular for $\mathbf{x}_0 = (0, y)^T$, with $\mathbf{G}(0, y) = \mathbf{O}$, and regular otherwise.

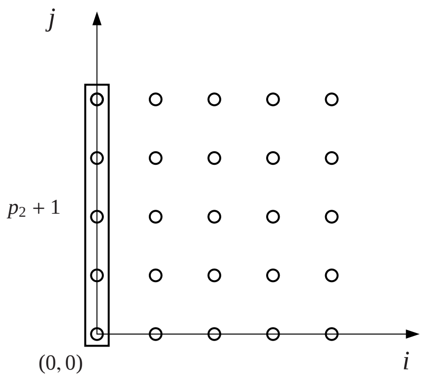

Figure 6: Index set $\mathbb{D}$ for Case I (collapsing edge) with $p_1 = p_2 = 4$

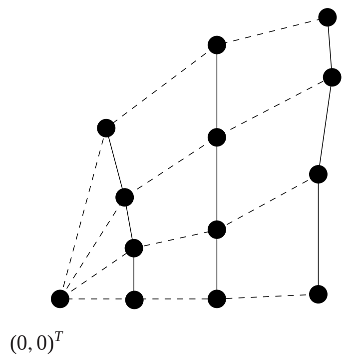

Figure 7: Control points for Case I

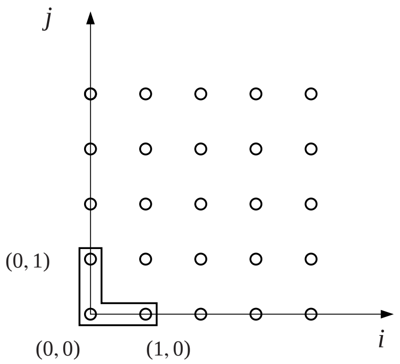

Figure 8: Index set $\mathbb{D}$ for Case II (collinear edges)

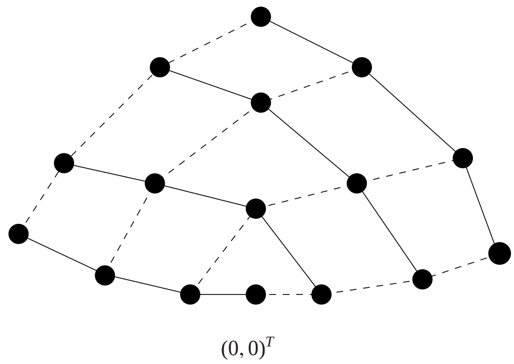

Figure 9: Control points for Case II

- **Case II: collinear edges.** Similar to Case I, let Ω be a B-spline patch of degree (p_1, p_2) consisting of $n_1.n_2$ tensor-product basis functions. The *index set of degeneration* $\mathbb{D}$ is defined as

  $$\mathbb{D} = \{(0,0), (1,0), (0,1)\}.$$

  The control points $\mathbf{P_j}$ are collinear for $\mathbf{j} \in \mathbb{D}$. The parameterization is singular for $\mathbf{x}_0 = (0,0)^T$, with $\mathbf{G}(0,0) = \mathbf{O}$, and regular elsewhere.

Remark 4.3 *Note that any tensor-product B-spline surface can be split into Bézier patches. Therefore results for basis functions on Bézier patches can be extended to more general domains with B-spline representations.*

An example of an index set for Case I is presented in Figure 6. The dots represent double indices $(i_1, i_2) \in \mathbb{I}$. The dots inside the bold-lined rectangle represent the set $\mathbb{D}$.

Figure 7 shows an example of a control point grid for a bivariate Bézier patch of degree $\mathbf{p} = (3,3)$. The control points that lie on a common thin continuous or dashed line have a common i_1- or i_2-index, respectively. This example is a valid Case I situation. Figure 8 visualizes the index sets $\mathbb{I}$ and $\mathbb{D}$ (bold continuous line) for a patch that belongs to Case II. Figure 9 shows a singular Bézier patch of degree $\mathbf{p} = (3,3)$. It shows the control point grid of an example of a Case II situation.

We will now analyze both cases separately and state regularity results for the test functions.

Theorem 4.4 *Let* $\mathbf{G}$ *be a tensor-product B-spline parameterization of degree* $\mathbf{p} = (p_1, p_2)$ *of the domain* Ω*. In Case I we define*

$$\mathbb{D}_1 = \{(i_1, i_2) \in \mathbb{I} : i_1 = 0\} \quad \textit{and} \quad \mathbb{D}_2 = \{(i_1, i_2) \in \mathbb{I} : i_1 \leq 1\}.$$

In Case II we have $\mathbb{D}_1 = \emptyset$. For $\mathbb{D}_2$ we consider two subcases. If the symmetry condition

$$\frac{\partial^2 \mathbf{G}}{\partial x^2}(0,0) = -\frac{\partial^2 \mathbf{G}}{\partial y^2}(0,0) \tag{8}$$

is fulfilled, then we choose

$$\mathbb{D}_2 = \{(i_1, i_2) \in \mathbb{I} : i_1 + i_2 \leq 2\} \setminus \{(1,1)\}.$$

Otherwise,

$$\mathbb{D}_2 = \{(i_1, i_2) \in \mathbb{I} : i_1 + i_2 \leq 2\}.$$

The test functions $\psi_{\mathbf{i}}$ fulfill $\psi_{\mathbf{i}} \notin H^1(\Omega)$ if and only if $\mathbf{i} \in \mathbb{D}_1$. Moreover, they satisfy $\psi_{\mathbf{i}} \notin H^2(\Omega)$ if and only if $\mathbf{i} \in \mathbb{D}_2$.

Proof. For the proof we restrict ourselves to Bézier parameterizations. This is sufficient as we pointed out in Remark 4.3. We will split the proof of the two statements into three parts. First we develop an approximation of the integrand in (5) or (7), respectively. This will be done using a Taylor expansion of the numerator and denominator of the integrands. Then we show the existence of the approximate integrals. Finally we conclude from that the existence of the original integrals.

We start with Case I and analyze the integral

$$\int_{\mathbf{B}} \left(\left| F_{\mathbf{i}}^{(1)} \times F_{\mathbf{i}}^{(2)} \right|^2 - (\det \nabla \mathbf{G})^2 \right) \frac{1}{J} \, \mathrm{d}\mathbf{x}$$

corresponding to the H^1-seminorm. In order to simplify the notation we will write $\mathbf{i} = (i_1, i_2) = (i, j)$ and $\mathbf{x} = (x, y)^T$. First we fix y and derive the Taylor expansions of

$$\left| F_{\mathbf{i}}^{(1)} \times F_{\mathbf{i}}^{(2)} \right|^2 - (\det \nabla \mathbf{G})^2$$

and $\det \nabla \mathbf{G}$ with respect to x around $x_0 = 0$. The assumptions made in Case I imply that

$$\mathbf{G}(x, y) = \sum_{i=1}^{p_1} \sum_{j=0}^{p_2} \mathbf{P}_{i,j} B_i(x) B_j(y)$$

where $B_i(x)$ and $B_j(y)$ are the Bernstein polynomials. Using $\phi_{i,j}(x, y) = B_i(x) B_j(y)$ we conclude

$$F_{\mathbf{i}}^{(1)} = \left(f_1(y) + \mathcal{O}(x), \, f_2(y) + \mathcal{O}(x), \, B_i'(x) B_j(y) \right)^T$$

and

$$F_{\mathbf{i}}^{(2)} = \left(x f_3(y) + \mathcal{O}\left(x^2\right), x f_3(y) + \mathcal{O}\left(x^2\right), B_i(x) B_j'(y) \right)^T,$$

where f_1, f_2, f_3 and f_4 are some linearly independent functions. Hence

$$\left| F_{\mathbf{i}}^{(1)} \times F_{\mathbf{i}}^{(2)} \right|^2 - (\det \nabla \mathbf{G})^2 = (B_i(x) g(y))^2 + \mathcal{O}(x)$$

for some function g. One can show easily that there exist constants $0 < C_1 < C_2$ such that

$$C_1 x \leq \det \nabla \mathbf{G}(x, y) \leq C_2 x$$

for all $(x, y)^T \in \mathbf{B}$. Hence there exist constants $0 < \underline{C} < \overline{C}$ such that

$$\underline{C} \int_{\mathbf{B}} (B_i(x) g(y))^2 \frac{1}{x} \, \mathrm{d}\mathbf{x} \quad \leq \quad \int_{\mathbf{B}} \left(\left| F_{\mathbf{i}}^{(1)} \times F_{\mathbf{i}}^{(2)} \right|^2 - (\det \nabla \mathbf{G})^2 \right) \frac{1}{J} \, \mathrm{d}\mathbf{x} \quad \leq \quad \overline{C} \int_{\mathbf{B}} (B_i(x) g(y))^2 \frac{1}{x} \, \mathrm{d}\mathbf{x}.$$

Since

$$\int_{\mathbf{B}} (B_i(x)\, g(y))^2 \frac{1}{x}\, \mathrm{d}\mathbf{x} < \infty$$

if and only if $i \geq 1$ the first statement follows immediately. Now we consider the H^2-seminorm integral

$$\int_{\mathbf{B}} \left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F^2 \frac{1}{J^5}\, \mathrm{d}\mathbf{x}.$$

Using a similar approach as for the H^1 integral we can show that

$$\left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F = x\, B_i(x)\, f(x,y)$$

where $f(x,y)$ is a function satisfying $C_1 < f(x,y) < C_2$, with constants $0 < C_1 < C_2$ for all x in a neighborhood of $x_0 = 0$. Hence there exist constants $0 < \underline{C} < \overline{C}$ such that

$$\underline{C} \int_{\mathbf{B}} (x\, B_i(x))^2 \frac{1}{x^5}\, \mathrm{d}\mathbf{x} \quad \leq \quad \int_{\mathbf{B}} \left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F^2 \frac{1}{J^5}\, \mathrm{d}\mathbf{x} \quad \leq \quad \overline{C} \int_{\mathbf{B}} (x\, B_i(x))^2 \frac{1}{x^5}\, \mathrm{d}\mathbf{x}.$$

Considering Case I, the second statement of the theorem follows since

$$\int_{\mathbf{B}} (B_i(x))^2 \frac{1}{x^3}\, \mathrm{d}\mathbf{x} < \infty$$

if and only if $i \geq 2$.

A similar strategy can be applied in Case II. As described in [14] there exist constants $0 < C_1 < C_2$ such that

$$C_1 (x+y) \leq \det \nabla \mathbf{G}(x,y) \leq C_2 (x+y)$$

for all $(x,y)^T \in \mathbf{B}$. Since all basis functions are bounded there exists a constant $0 < \overline{C}$ such that

$$\int_{\mathbf{B}} \left(\left| F_{\mathbf{i}}^{(1)} \times F_{\mathbf{i}}^{(2)} \right|^2 - (\det \nabla \mathbf{G})^2 \right) \frac{1}{J}\, \mathrm{d}\mathbf{x} \quad \leq \quad \overline{C} \int_{\mathbf{B}} \frac{1}{x+y}\, \mathrm{d}\mathbf{x}$$

for all $\mathbf{i}$. The integral of $1/(x+y)$ is bounded in any case. Now we analyze

$$\int_{\mathbf{B}} \left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F^2 \frac{1}{J^5}\, \mathrm{d}\mathbf{x},$$

where $\mathcal{B}$ depends on the index $\mathbf{i}$ as in (6). One can show that for $\mathbf{i} = (i,j)$ with $i + j \geq 3$ there exists a $C > 0$ such that

$$\left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F \leq C (x+y)^{i+j-1}.$$

If the symmetry condition (8) is not fulfilled, then there exists a constant $0 < \underline{C}$ such that

$$\underline{C}\, (x+y)^{\max\{i+j-1,0\}} \leq \left\| \mathrm{Cof}(\nabla \mathbf{G})\, \mathcal{B}\, \mathrm{Cof}(\nabla \mathbf{G})^T \right\|_F$$

for $i + j \leq 2$. If condition (8) is fulfilled and $(i,j) \neq (1,1)$ then this bound is still valid.

If we omit the case $i = j = 1$ (under condition (8)) we conclude

$$\underline{C} \int_{\mathbf{B}} \frac{1}{(x+y)^4}\, \mathrm{d}\mathbf{x} \leq |\psi_{\mathbf{i}}|^2_{H^2(\Omega)}$$

for $i + j \leq 2$ and

$$|\psi_{\mathbf{i}}|^2_{H^2(\Omega)} \leq \overline{C} \int_{\mathbf{B}} (x+y)^{2(i+j)-7}\, \mathrm{d}\mathbf{x}$$

for $i + j \geq 3$. Since

$$\int_{\mathbf{B}} (x+y)^k \, \mathrm{d}\mathbf{x} < \infty$$

if and only if $k \geq -1$ the statement follows immediately.

The only remaining case is $i = j = 1$ and condition (8) being valid. For this configuration the lower degree terms cancel out and we get

$$\left\| \mathrm{Cof}\,(\nabla \mathbf{G})\ \mathcal{B}\, \mathrm{Cof}\,(\nabla \mathbf{G})^T \right\|_F \leq C\,(x+y)^2$$

for some $C > 0$. Hence we get

$$\left| \psi_{(1,1)} \right|^2_{H^2(\Omega)} \leq \overline{C} \int_{\mathbf{B}} \frac{1}{x+y} \, \mathrm{d}\mathbf{x} < \infty$$

which concludes the proof. □

Summing up, this theorem states that test functions corresponding to control points that are close to the singularity are not sufficiently regular.

4.2. *Modified test functions*

It turns out that certain linear combinations of the test functions are sufficiently regular. We present such a modification scheme.

Theorem 4.5 *Consider again the assumptions of Theorem 4.4, and let* $\mathbf{P_i} = \left(P^1_{\mathbf{i}}, P^2_{\mathbf{i}}\right)^T$ *be the control points of the parameterization. Let* $\mathbb{D}_2$ *be the set defined in Theorem 4.4. The set* $\mathcal{V}_h$ *is the space of tensor-product test functions. The function space* $\hat{\mathcal{V}}_h$ *is defined as the span of*

$$\begin{aligned}
\hat{\psi}_{0,0}(\boldsymbol{\xi}) &= \sum_{\mathbf{i}\in\mathbb{D}_2} C_{\mathbf{i}}\ \psi_{\mathbf{i}}(\boldsymbol{\xi}), \\
\hat{\psi}_{1,0}(\boldsymbol{\xi}) &= \sum_{\mathbf{i}\in\mathbb{D}_2} \hat{P}^1_{\mathbf{i}}/\hat{P}_{\max}\ \psi_{\mathbf{i}}(\boldsymbol{\xi}), \\
\hat{\psi}_{0,1}(\boldsymbol{\xi}) &= \sum_{\mathbf{i}\in\mathbb{D}_2} \hat{P}^2_{\mathbf{i}}/\hat{P}_{\max}\ \psi_{\mathbf{i}}(\boldsymbol{\xi}), \quad \textit{and} \\
\hat{\psi}_{\mathbf{i}}(\boldsymbol{\xi}) &= \phi_{\mathbf{i}}\left(\mathbf{G}^{-1}(\boldsymbol{\xi})\right) \quad \textit{for } \mathbf{i} \in \mathbb{I}\backslash\mathbb{D}_2,
\end{aligned}$$

where

$$\hat{P}^k_{\mathbf{i}} = \frac{P^k_{\mathbf{i}} - \min_{\mathbf{j}\in\mathbb{D}_2}\left\{P^k_{\mathbf{j}}\right\}}{\max_{\mathbf{j}\in\mathbb{D}_2}\left\{P^k_{\mathbf{j}}\right\} - \min_{\mathbf{j}\in\mathbb{D}_2}\left\{P^k_{\mathbf{j}}\right\}} \quad \textit{and} \quad C_{\mathbf{i}} = 1 - \frac{\hat{P}^1_{\mathbf{i}} + \hat{P}^2_{\mathbf{i}}}{\hat{P}_{\max}}$$

with $\hat{P}_{\max} = \max_{\mathbf{j}\in\mathbb{D}_2}\left\{\hat{P}^1_{\mathbf{j}} + \hat{P}^2_{\mathbf{j}}\right\}$. *Under these conditions we obtain* $\hat{\mathcal{V}}_h \subseteq \mathcal{V}_h \cap H^2(\Omega)$.

Proof. The proof of this theorem is a simple consequence of Theorem 4.4, similar to the proof of Theorem 3.4. □

The newly defined test functions $\hat{\psi}_{1,0}(\boldsymbol{\xi})$, $\hat{\psi}_{0,1}(\boldsymbol{\xi})$ and $\hat{\psi}_{0,0}(\boldsymbol{\xi})$ can be seen as local reconstructions of the coordinate functions $c_1(\boldsymbol{\xi}) = \xi_1$, $c_2(\boldsymbol{\xi}) = \xi_2$ and $c(\boldsymbol{\xi}) = 1 - \xi_1 - \xi_2$, respectively. Note that the reconstructed test functions in $\hat{\mathcal{V}}_h$ still maintain the desired properties like non-negativity and the partition of unity. To demonstrate the presented modification scheme we will discuss two examples. The first example belongs to Case I.

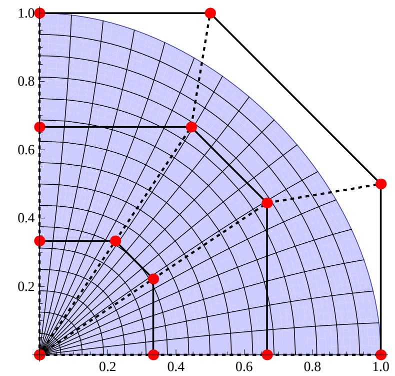

Figure 10: Control points for Example 4.6

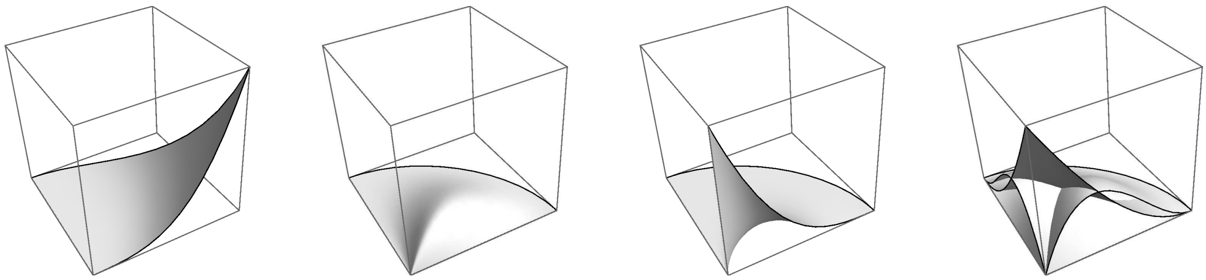

Figure 11: Test functions $\psi_{3,0}$, $\psi_{1,1}$, $\psi_{0,0}$ (3 plots on the left) and test functions $\hat{\psi}_{0,0}$, $\hat{\psi}_{1,0}$, $\hat{\psi}_{0,1}$ (right plot) for Example 4.6

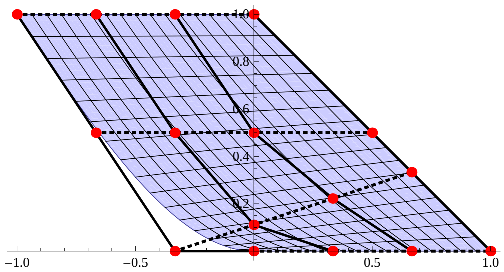

Figure 12: Control points for Example 4.7

Example 4.6 We consider a Bézier patch of degree $\mathbf{p} = (3, 3)$ and control points as shown in Figure 10.

Four control points coincide and lie in the origin, causing a singularity. In this example we have test functions $\psi_{i,j}(\xi)$ for $0 \leq i, j \leq 3$. Theorem 4.4 states that the test functions $\psi_{0,j}$ are not in $H^1(\Omega)$ and that the test functions $\psi_{1,j}$ are in $H^1(\Omega)$ but not in $H^2(\Omega)$. Nevertheless, Theorem 4.5 states that we can construct alternative test functions to replace the ones which are not sufficiently regular. Figure 11 depicts examples of test functions.

The three left figures show the function $\psi_{3,0}$ which fulfills $\psi_{3,0} \in H^1$, the function $\psi_{1,1}$, with $\psi_{1,1} \in H^1$ and $\psi_{1,1} \notin H^2$, and the function $\psi_{0,0}$, with $\psi_{0,0} \notin H^1$. The rightmost figure shows the test functions $\hat{\psi}_{0,0}$, $\hat{\psi}_{1,0}$ and $\hat{\psi}_{0,1}$ as defined in Theorem 4.5. All functions $\hat{\psi}_{i,j}$ are in H^2.

In the next example we consider a parameterization fulfilling the Assumption of Case II.

Example 4.7 We consider a Bézier patch of degree $\mathbf{p} = (3, 3)$ and control points as in Figure 12.

Similar to Example 4.6 Figure 13 shows examples of test functions.

Here we have that $\psi_{3,3}$ is in H^2, $\psi_{1,1}$ and $\psi_{0,0}$ are in H^1 but not in H^2 and the functions $\hat{\psi}_{0,0}$, $\hat{\psi}_{1,0}$ and $\hat{\psi}_{0,1}$ as defined in 4.5 are in H^2.

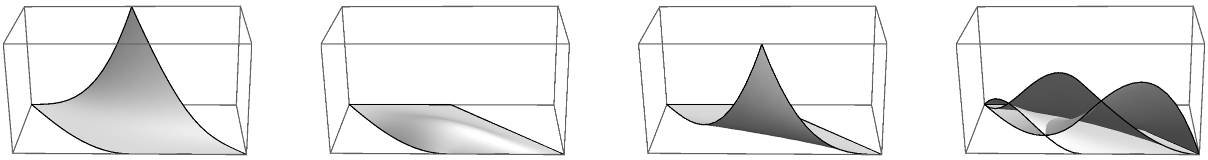

Figure 13: Test functions $\psi_{3,3}$, $\psi_{1,1}$, $\psi_{0,0}$ (3 plots on the left) and test functions $\hat{\psi}_{0,0}$, $\hat{\psi}_{1,0}$, $\hat{\psi}_{0,1}$ (right plot) for Example 4.7

In both examples we get similar results that can be extended to general B-spline parameterizations. Another example of singular patches are fillet patches (see e.g. [28]). In that case the singularity is caused by a 0 degree angle in contrast to the 180 degree angle of case II. These patches can be used to represent sharp cusps with parallel tangents. The results developed in this paper do not cover this type of singularity but the theory can be adapted to it.

5. Structurally equivalent parameterizations and sweeping

We introduce a framework to derive regularity results for more general parameterizations.

5.1. *Structurally equivalent parameterizations*

In higher dimensions it becomes very technical to prove regularity results for singular parameterizations. However, its relatively easy to derive results if the general parameterization is *structurally equivalent* to a reference parameterization where regularity results are available. The following definition is used to describe such an equivalence.

Definition 5.1 *Two parameterizations $\hat{\mathbf{G}}$ and $\mathbf{G}$ are said to be* structurally equivalent *of order k if $\hat{\mathbf{G}} \circ \mathbf{G}^{-1} \in C^k$ and $\mathbf{G} \circ \hat{\mathbf{G}}^{-1} \in C^k$ where all derivatives are bounded.*

It is possible to derive conditions on the control points and weights of a B-spline parameterization which imply this property.

Note that this notion of structural equivalence is different from the notion used in [14]. First, it also considers higher – and not only first – derivatives. Second, the derivatives have to be bounded, while the notion in [14] requires the eigenvalues of the Jacobian to be bounded.

The following result is an immediate consequence of this definition.

Proposition 5.2 *If two parameterizations $\hat{\mathbf{G}}$ (with test functions $\hat{\psi}_{\mathbf{i}}$ on $\hat{\Omega}$) and $\mathbf{G}$ (with test functions $\psi_{\mathbf{i}}$ on Ω) with common basis functions $\phi_{\mathbf{i}}$ on Ω_0 and common index set $\mathbb{I}$ are structurally equivalent of order k, then $\psi_{\mathbf{i}} \in H^k(\Omega)$ if and only if $\hat{\psi}_{\mathbf{i}} \in H^k\left(\hat{\Omega}\right)$.*

We will omit the (simple) proof of this proposition. In the next section we will use the definition of structurally equivalent parameterizations and Proposition 5.2 to prove regularity results for several examples.

5.2. *Swept parameterizations*

In this chapter we will present special 3-dimensional domains which are derived from lower dimensional domains. Let $\mathbf{G}^{[3]}$ be the parameterization of the 3-dimensional domain $\Omega^{[3]}$ having basis functions $\left(\phi_{(i,j)}(x,y)\,\phi_k(z)\right)_{(i,j,k)\in\mathbb{I}}$, control points $(\mathbf{P_i})_{\mathbf{i}\in\mathbb{I}}$ and the index set

$$\mathbb{I} = \left\{\mathbf{i} = (i,j,k) \in \mathbb{Z}^3 : \mathbf{0} \le \mathbf{i} \le (n_1,n_2,n_3) - \mathbf{1}\right\}.$$

The two-dimensional domain $\Omega^{[2]}$ has the parameterization $\mathbf{G}^{[2]}$ with basis functions $\left(\phi_{\mathbf{j}}(x,y)\right)_{\mathbf{j}\in\mathbb{J}}$, control points $\left(\mathbf{Q_j}\right)_{\mathbf{j}\in\mathbb{J}}$ and the index set

$$\mathbb{J} = \left\{\mathbf{j} = (i,j) \in \mathbb{Z}^2 : \mathbf{0} \le \mathbf{j} \le (n_1,n_2) - \mathbf{1}\right\}.$$

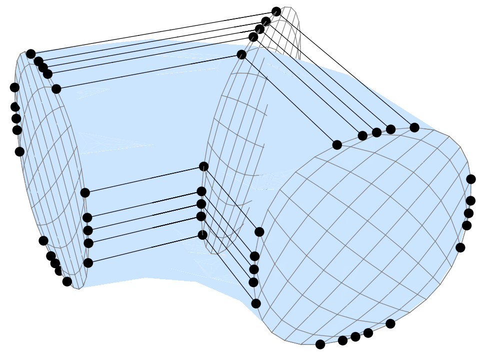

Figure 14: Quarter of a torus and control point grid

Now we can state the following theorem for swept volume parameterizations (similar to a result in [14]).

Lemma 5.3 *Let $\Omega^{[3]}$ be a volume constructed from the two-dimensional domain $\Omega^{[2]}$, i.e. for $\mathbf{i} \in \mathbb{I}$ the control point $\mathbf{P_i}$ fulfills*

$$\mathbf{P}_{(i,j,k)} = \left(Q^1_{(i,j)}, Q^2_{(i,j)}, P_k\right)^T , \tag{9}$$

where $(P_k)_{k\in\{0,\ldots,n_3-1\}}$ is a strictly monotonically increasing sequence. Each trivariate test function $\psi_{(i,j,k)}$ fulfills

$$\psi_{(i,j,k)} = \phi_{(i,j)}\phi_k \circ \left(\mathbf{G}^{[3]}\right)^{-1} \in H^1\left(\Omega^{[3]}\right)$$

if and only if the bivariate test function $\psi_{(i,j)}$ fulfills

$$\psi_{(i,j)} = \phi_{(i,j)} \circ \left(\mathbf{G}^{[2]}\right)^{-1} \in H^1\left(\Omega^{[2]}\right).$$

This theorem states existence results for prismatic or cylindrical domains. It can now be used to cover more general domains using Proposition 5.2.

Example 5.4 Figure 14 shows the quarter of a torus. The parameterization of the torus is structurally equivalent to the cylindrical parameterization shown in Figure 14 of [14].

For this example all test functions on the torus are in H^1. In Figure 14 we present a control point grid and mark especially those control points corresponding to test functions that are not in H^2 (black dots). In this picture not the entire control grid is plotted, but only parts thereof.

The total number of control points for this example is $10 \times 10 \times 3$, hence the dimension of the function space $\mathcal{V}_h$ is 300. Each quintuple of test functions, corresponding to the control points depicted in Figure 14, is not in H^2. According to our modification scheme one can recover 3 sufficiently regular test functions out of each quintuple. Since there are 12 such groups of control points we lose 12×5 degrees of freedom but regain 12×3 via the modification scheme. Hence the modified function space $\hat{\mathcal{V}}_h$ has 276 degrees of freedom.

The considered class of three dimensional domains that is covered by the presented theory is by far too small to cover all interesting cases. It is of particular interest to develop a similar theory for more general spatial domains with singular parameterizations, like cones or volumes with a smooth boundary (e.g. a sphere).

6. Conclusions

In this paper we considered the isogeometric method to solve partial differential equations on 1-, 2- and 3-dimensional domains. We specifically analyzed situations where the parameterization of the domain contains singularities. Such degeneracies can be caused by collapsing control points or by control points that are collinear at the boundary, and they are highly useful for compactly representing technically interesting geometries.

First we treated the 1-dimensional case where we assumed that the first α control points collapse. In that case we could show that the first $\lfloor \alpha/2 \rfloor + 1$ test functions are not in H^1 and that the first $\lfloor (1+3\alpha)/2 \rfloor + 1$ test functions are not in H^2. This behavior is remarkable since not only those test functions corresponding to degenerating control points are affected but also neighboring ones. Similar results can be shown for 2-dimensional domains, where we treated two special cases separately.

Further, we presented a modification scheme for all cases to regain the needed regularity properties. We could show that specific linear combinations of test functions are sufficiently regular. The presented schemes lead to convenient discrete function spaces which seem fruitful for future analysis, e.g. approximation properties.

The presented results can be extended to parameterizations with several singular points, provided that the singularities occur at the vertices of the polynomial or rational segments. More general situations, like singularities appearing in the interior of patches, are not yet covered. This remains an objective for future research.

Some of the main targets for further analysis are approximation properties on singular domains and quantitative results concerning the stiffness matrix of a variational problem. The extension to higher dimensions is also of interest, since we only considered swept parameterizations so far.

Appendix. Proof of Lemma 2.2

During the proof we will omit the index $\mathbf{i}$, in order to improve the readability. The chain rule applied to

$$\psi\left(\mathbf{G}\left(\mathbf{x}\right)\right) = \phi\left(\mathbf{x}\right)$$

leads to

$$\frac{\partial \phi}{\partial x_i} = \sum_{m=1}^{d} \frac{\partial \psi}{\partial \xi_m} \frac{\partial \mathbf{G}_m}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_j \partial x_i} = \sum_{m,n=1}^{d} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_m} \frac{\partial \mathbf{G}_m}{\partial x_i} \frac{\partial \mathbf{G}_n}{\partial x_j} + \sum_{m=1}^{d} \frac{\partial \psi}{\partial \xi_m} \frac{\partial^2 \mathbf{G}_m}{\partial x_j \partial x_i}.$$

We have $\mathrm{Cof} A = 1$ for a scalar A and

$$\mathrm{Cof} \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} A_{2,2} & -A_{2,1} \\ -A_{1,2} & A_{1,1} \end{pmatrix}$$

for a 2×2 matrix $\left(A_{i,j}\right)_{i,j=1}^{2}$. Since

$$A^{-T} = \frac{1}{\det A} \mathrm{Cof} A$$

we conclude

$$\frac{\partial \psi}{\partial \xi_i} = \sum_{k=1}^{d} C_{m,k} \frac{\partial \phi}{\partial x_k} \frac{1}{J}.$$

Hence

$$\frac{\partial^2 \phi}{\partial x_j \partial x_i} - \sum_{k,l=1}^{d} C_{l,k} \frac{\partial \phi}{\partial x_k} \frac{\partial^2 \mathbf{G}_l}{\partial x_j \partial x_i} \frac{1}{J} = \sum_{m,n=1}^{d} \frac{\partial^2 \psi}{\partial \xi_n \partial \xi_m} \frac{\partial \mathbf{G}_m}{\partial x_i} \frac{\partial \mathbf{G}_n}{\partial x_j},$$

which leads to

$$\frac{\partial^2\psi}{\partial\xi_n\partial\xi_m} = \frac{1}{J^3}\sum_{i,j=1}^{d} C_{i,m}C_{j,n}\left(\frac{\partial^2\phi}{\partial x_j\partial x_i}J - \sum_{k,l=1}^{d} C_{l,k}\frac{\partial\phi}{\partial x_k}\frac{\partial^2\mathbf{G}_l}{\partial x_j\partial x_i}\right).$$

Finally we arrive at

$$|\psi|^2_{H^2(\Omega)} = \int_\Omega \sum_{m,n=1}^{d}\left(\frac{\partial^2\psi}{\partial\xi_n\partial\xi_m}\right)^2 \mathrm{d}\boldsymbol{\xi} = \int_{\mathbf{B}} \sum_{m,n=1}^{d}\left(\frac{N_{m,n}}{J^3}\right)^2 J\,\mathrm{d}\mathbf{x},$$

which concludes the proof. □

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