CHARACTERIZATION OF BIVARIATE HIERARCHICAL QUARTIC BOX SPLINES ON A THREE-DIRECTIONAL GRID

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Abstract. We consider the adaptive refinement of bivariate quartic C^2-smooth box-spline spaces on the three-directional (type-I) grid G. The polynomial segments of these box splines belong to a certain subspace of the space of quartic polynomials, which will be called the space of special quartics. Given a finite sequence $(G^\ell)_{\ell=0,\dots,N}$ of dyadically refined grids, we obtain a hierarchical grid by selecting cells from each level such that their closure covers the entire domain Ω, which is a bounded subset of $\mathbb{R}^2$. A suitable selection procedure allows to define a basis spanning a hierarchical box spline space. As our main result, we derive a characterization of this space. More precisely, under certain mild assumptions on hierarchical grid, the hierarchical spline space is shown to contain all C^2-smooth functions whose restrictions to the cells of the hierarchical grid are special quartic polynomials.

1. Introduction

Box splines and the functions contained in the spaces spanned by them form a highly interesting class of piecewise polynomial functions, which are defined on regular grids. A comprehensive introduction to this topic is given in the monograph [3]. Box splines possess a number of useful theoretical and practical properties that make them well-suited for applications. It has been shown that

- Box splines have small support (the union of few cells of the underlying grid),
- they are positive in the interior of its support, and
- box splines are refinable, i.e., the box spline spaces on (e.g.) dyadically refined grids are nested [1, 4].

Moreover, a substantial number of results on the approximation power of box splines is described in the literature, e.g. [11, 14].

In this paper we shall restrict ourselves to the case of C^2-smooth quartic box-splines on a type-I triangulation of $\mathbb{R}^2$, cf. Fig. 1. These functions form the mathematical basis of Loop's subdivision scheme

and are therefore used to construct the regular parts of the corresponding subdivision surfaces, cf. [10, 15].

On each triangular cell, the space generated by these basis functions spans a 12-dimensional subspace of the space of quartic bivariate polynomials. This subspace, which will be called the space of special quartics, is known to contain the cubic polynomials. Moreover, it is known that any locally supported function in the underlying spline space can be represented as a linear combination of these box splines [2].

Local refinement of box spline spaces is not automatically supported, hence a hierarchical approach should be used to obtain this property. Several recent publications explore box splines in a hierarchical setting. In [7] quadratic and cubic hierarchical box splines are studied, and applied to surface fitting and for the numerical solution of partial differential equations. Hierarchical ZP elements were also studied in [16].

In the present paper we consider the adaptive refinement of bivariate quartic C^2-smooth box-spline spaces on the three-directional (type-I) grid G. More precisely, given a finite sequence $(G^\ell)_{\ell=0,\dots,N}$ of dyadically refined grids, we obtain a hierarchical grid by selecting cells from each level such that their closure covers the entire domain Ω, which is a bounded subset of $\mathbb{R}^2$. Using a suitable selection procedure, which generalizes the hierarchical B-spline basis introduced by Kraft [8] to quartic C^2 box splines, we define a basis spanning a hierarchical box spline space. As our main result, we characterize the span of this space as the space containing all C^2-smooth functions whose restrictions to the cells of the hierarchical grid are special quartic polynomials. Our derivations are based on the approach in [13], which has been modified suitably to deal with the box spline case.

2. Preliminaries

2.1. **Bivariate splines on regular grids.** We consider bivariate splines on a three-directional grid in the plane $\mathbb{R}^2$, see Fig. 1. Let us denote by $\mathcal{P}_d$ the linear space of polynomials in $\mathbb{R}[x, y]$ of bidegree less than or equal to d.

We consider a partition G_Ω of a polygonal domain $\Omega \subset \mathbb{R}^2$ into mutually disjoint cells, where each cell is an open set and the closure of the union of all cells equals Ω. In addition we choose a linear space T of functions on $\mathbb{R}^2$. A typical choice would be $T = \mathcal{P}_d$, but other choices are also possible.

Let $S^r(G_\Omega, T)$ be the space of C^r splines is defined on G_Ω,

$$S^r(G_\Omega, T) = \{s \in C^r(\Omega) : s|_\triangle \in T|_\triangle \text{ for all cells } \triangle \in G_\Omega\}.$$

This definition is quite general and applies to any partition of any planar domain in $\mathbb{R}^2$. Throughout this paper, we consider a special triangulation which allows to construct splines with particularly nice properties.

More precisely, we consider the bi-infinite grid in $\mathbb{R}^2$ with lines $\mathbb{R} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{R}$ and the triangulation obtained by adding the north-east diagonals in the squares of the bi-infinite grid, see Fig. 1. This produces a three-directional grid which we denote by G. The grid G is a set which contains the elementary triangles (which are called cells) as its elements, where each of the triangles is considered as an open subset of $\mathbb{R}^2$.

This type of grid is called called a type-I triangulation in the literature [9]. The spline spaces on triangulations of this type have been studied thoroughly in the rich literature on this subject. In particular, they include box-spline spaces, which are interesting due to their elegant construction and simple refinement algorithm.

All results concerning splines on type-I triangulation remain valid under affine transformations of the underlying grid G. For instance, these transformations include scalings of the grid (and we will use this fact later when constructing hierarchical spline spaces), but also affine mappings that transform all triangles into equilateral ones, which reveals the built-in symmetries of these spline spaces.

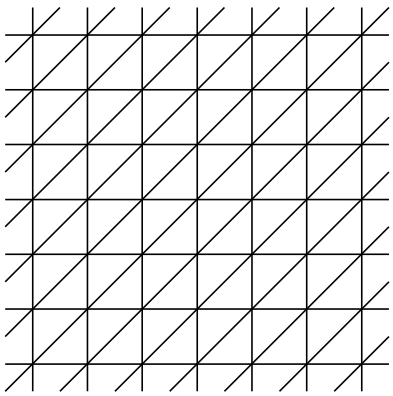

FIGURE 1. Three-directional grid G.

2.2. **Quartic box splines.** We restrict ourselves to polynomials $\mathcal{P}_4$ of degree up to four and we will denote this space simply by $\mathcal{P}$. For each triangle $\triangle \in G$, let us denote by $\mathcal{P}|_{\triangle}$ the linear space formed by the

restrictions $f|_{\triangle}$ of the polynomials $f \in \mathcal{P}$ to $\triangle$, i.e.,

$$\mathcal{P}|_{\triangle} = \{f|_{\triangle} : f \in \mathcal{P}\}.$$

For a given triangle $\triangle$, any bivariate polynomial can be represented as a linear combination of the associated bivariate Bernstein polynomials on this triangle [9],

$$f|_{\triangle} = \sum_{i+j+k=4} c_{ijk} B^4_{ijk}. \tag{1}$$

with real coefficients c_{ijk}. Each Bernstein polynomial B^4_{ijk} has an associated Greville point, which possesses the barycentric coordinates $(i/4, j/4, k/4)$ with respect to the triangle. This representation of the polynomials is quite useful for the efficient evaluation of the functions and their derivatives at a given point.

The coefficients in Fig. 2, which are placed at the Greville points, define a piecewise polynomial function, whose support is the set of these triangles. The multiple 1/24 of this function is the special box spline $N_{2,2,2}$ on the three-direction grid. It has polynomial degree 4 and is C^2 smooth. This box spline is our main object of interest and will be denoted by $\mathcal{B}$. Note that this box spline forms the mathematical basis of Loop subdivision surfaces [10, 15].

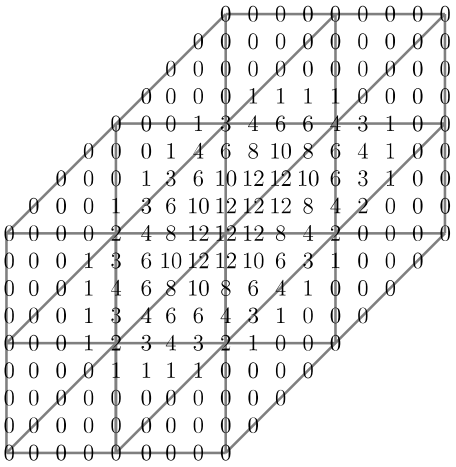

FIGURE 2. Support and the Bernstein coefficients of the scaled box spline $24\mathcal{B}$. The central vertex is located at the origin.

The translates

$$\beta_{ij}(\cdot) = \mathcal{B}(\cdot - (i, j)), \quad (i, j) \in \mathbb{Z}^2$$

are known to form a locally linearly independent set

$$B = \{\beta_{ij} \,:\, (i,j) \in \mathbb{Z}^2\}. \tag{2}$$

in the following sense: for any open set A, the translates

$$B_A = \{\beta_{ij} \in B : \operatorname{supp}(\beta_{ij}) \cap A \neq \emptyset\}$$

restricted to A are linearly independent [9]. Here $\operatorname{supp}(f)$ denotes the support of the function f.

2.3. **Contact of polynomial pieces.** By construction, each function β_{ij} is associated with the lattice point (i,j). For a cell $\triangle$ in G, let $\bar{\triangle}$ denote the closure of $\triangle$. We consider the translates whose support contains the given cell $\triangle$,

$$B_\triangle = \{\beta_{ij} : \operatorname{supp}(\beta_{ij}) \cap \triangle \neq \emptyset\}.$$

This set is formed by the 12 translates β_{ij}, which are associated with the vertices of the 1-ring neighborhood of $\triangle$ in the three directional grid, see Fig. 3.

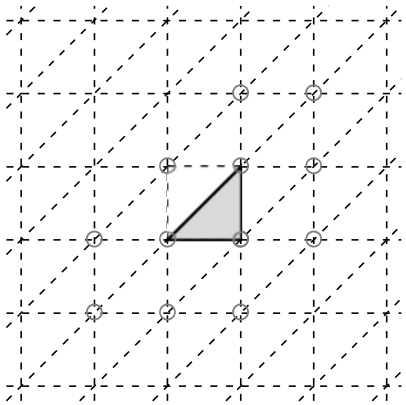

FIGURE 3. The 1-ring of a triangle.

We now consider the linear space spanned by the restrictions of these translates to the given triangle,

$$\mathbb{V}_\triangle = (\operatorname{span} B_\triangle)|_\triangle. \tag{3}$$

Since this space is a subset of the space of quartic polynomials (since its dimension is 12 only), we will call it the space of special quartics on $\triangle$.

Remark 1. It can be shown that $\mathbb{V}_{\triangle} = \hat{\mathcal{P}}|_{\triangle}$ where

$$\hat{\mathcal{P}} = \text{span}\,(\mathcal{P}_3 \cup \{x^4 - 2x^3y, y^4 - 2xy^3\}).$$

In particular, it should be noted that $\hat{\mathcal{P}}$ is independent of the chosen cell $\triangle$. The proof of this observation is postponed to the Appendix. $\diamond$

Any polynomial $f|_{\triangle} \in \mathbb{V}_{\triangle}$ has a unique representation

$$f|_{\triangle}(\mathbf{x}) = \sum_{\beta \in B_{\triangle}} \lambda_{\triangle}^{\beta}(f|_{\triangle})\, \beta(\mathbf{x}), \quad \mathbf{x} \in \triangle,$$

with certain coefficients $\lambda_{\triangle}^{\beta}(f|_{\triangle}) \in \mathbb{R}$.

Now we consider two cells $\triangle$ and $\triangle'$ which share an edge. The 1-rings around $\triangle$ and $\triangle'$ have 10 vertices in common, and hence $B_{\triangle}$, and $B_{\triangle'}$ share 10 elements, see Fig. 4.

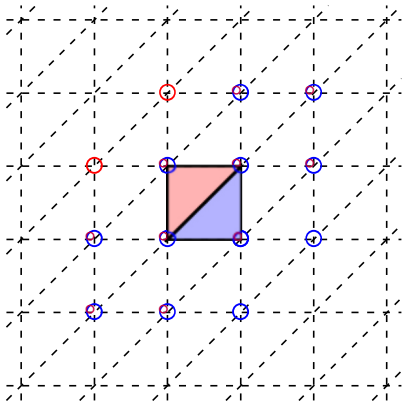

FIGURE 4. Active box splines on a square domain consisting of two triangles. Twelve functions are active (non-zero) on each triangle. Exactly ten of those are active on both triangles.

The following notion of contact will be important in the sequel for the definition of spline spaces.

Definition 2. We say that two polynomials $f|_{\triangle} \in \mathcal{P}_{\triangle}$ and $f'|_{\triangle'} \in \mathcal{P}_{\triangle'}$ have contact of order 2 (and write $f|_{\triangle} \sim f'|_{\triangle'}$), if

$$\forall \mathbf{x} \in \bar{\triangle} \cap \bar{\triangle}', \quad \frac{\partial^{i+j}}{\partial x^i \partial y^j} f|_{\triangle}(\mathbf{x}) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f'|_{\triangle'}(\mathbf{x}) \tag{4}$$

for all i and j with $0 \le i + j \le 2$. The derivatives and values at points on a triangle boundary are obtained by one-sided limits.

Lemma 3 (Contact Characterization Lemma (CCL)). Let $f|_\triangle \in \mathbb{V}_\triangle$ and $f'|_{\triangle'} \in \mathbb{V}_{\triangle'}$ on two disjoint triangles $\triangle, \triangle'$, and assume that the two triangles have a common edge $e = \bar{\triangle} \cap \bar{\triangle}'$. Then the two polynomials $f|_\triangle$ and $f'|_{\triangle'}$ have a contact of order 2 if and only if

$$\forall\ \beta \in B\ :\ \beta|_e \neq 0|_e \implies \lambda^\beta_\triangle(f|_\triangle) = \lambda^\beta_{\triangle'}(f'|_{\triangle'}).$$

Proof. Let $\triangle = (v_1, v_2, v_3)$ and $\triangle' = (v_4, v_3, v_2)$ be two triangles in G sharing the edge $e = (v_2, v_3)$. The two polynomials $f|_\triangle$ and $f'|_{\triangle'}$ have C^2 contact on the edge e if and only if their Bernstein coefficients c_{ijk} and c'_{ijk} with respect to the triangles $\triangle$, and $\triangle'$ respectively, satisfy the conditions

$$c'_{njk} = \sum_{\nu+\mu+\kappa=n} c_{\nu,k+\mu,j+\kappa} B^n_{\nu\mu\kappa}(v_4)\ , \qquad \begin{matrix} j+k = 4-n \\ n = 0,1,2. \end{matrix} \tag{5}$$

This establishes 12 conditions on the coefficients c_{ijk} and c'_{ijk} of $f|_\triangle$ and $f'|_{\triangle'}$.

On the other hand, since $f|_\triangle \in \mathbb{V}_\triangle$ and $f'|_{\triangle'} \in \mathbb{V}_{\triangle'}$, then $f|_\triangle$ and $f'|_{\triangle'}$ can be expressed as

$$f|_\triangle = \sum_{i=1}^{12} \lambda^{\beta_i}_\triangle(f|_\triangle)\beta_i|_\triangle \quad \text{and} \quad f'|_{\triangle'} = \sum_{i=1}^{12} \lambda^{\beta_i}_{\triangle'}(f'|_{\triangle'})\beta'_i|_{\triangle'},$$

where β_i and β'_i run over the active box splines in $\triangle$ or $\triangle'$, respectively. The restriction of the translates of the box spline $\beta_i|_\triangle$ and $\beta'_i|_{\triangle'}$ can be written as linear combinations of the Bernstein bases $B^n_{\nu\mu\kappa}$ and $B'^n_{\nu\mu\kappa}$ associated to the triangles $\triangle$ and $\triangle'$, respectively. This yields an expression of $f|_\triangle$ and $f'|_{\triangle'}$ as linear combinations of the respective Bernstein basis and whose coefficients c_{ijk} and c'_{ijk} are linear combinations in the $\lambda^{\beta_i}_\triangle$ and $\lambda^{\beta_i}_{\triangle'}$, $i = 1, \ldots, 12$, respectively. These coefficients must satisfy the relations (5). Solving the system yields the conditions stated in the lemma. □

3. Special quartic splines on multi-cell domains

3.1. **Piecewise polynomial functions on triangular domains.** In the three-directional grid G, we will consider finite set of cells (triangles) $M \subset G$. Any such set M corresponds to a bounded domain

$$\mathcal{M} = \bigcup_{\triangle \in M} \triangle = \bigcup M \subset \mathbb{R}^2. \tag{6}$$

If the domain $\mathcal{M}$ is connected, then we will say that the set M of triangles is also connected. We need a stronger version of connectivity which excludes vertex-vertex contacts of triangles.

Definition 4. A set M of triangles is said to be $\star$-connected if it is connected and if additionally for any two triangles $\triangle$, $\triangle'$ in M, which have a common vertex v, $\bar{\triangle} \cap \bar{\triangle}' \supseteq \{v\}$, there is a chain of triangles $\triangle_0, \triangle_1, \ldots, \triangle_m$ all in M such that $\triangle_0 = \triangle$, $\triangle_m = \triangle'$ and $\bar{\triangle}_i \cap \bar{\triangle}_{i+1} = e_i$ for some edge $e_i \in M$ that contains the vertex v, $v \in e_i$, for $i = 0, \ldots, m-1$.

In particular, two triangles are $\star$-connected triangles if they possess a common edge.

We require the following condition.

Condition 5. The set M of triangles is assumed to be a union of finitely many mutually disconnected finite sets of triangles, each of which is $\star$-connected.

In particular, this condition on the set of triangles M implies that we do not allow "kissing vertices" in any connected component of $\mathcal{M}$; or in other words, M is a triangulation of a 2-manifold $\mathcal{M}$ with boundary.

Example 6. The domain $\mathcal{M} = \bigcup M$ in Fig. 5 has only one component and it is not $\star$-connected. In Fig. 6, the set of triangles M is modified in several ways by adding and deleting triangles and the different components (when more than one) are all $\star$-connected, and hence the domains satisfy Condition 5. ◇

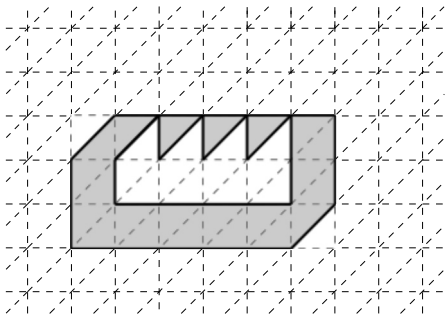

FIGURE 5. Example of a set M which is connected but not $\star$-connected.

In the remainder of the paper, every set M is assumed to satisfy Condition 5. The set of the translates of box splines that act on the triangles M will be denoted as

$$B_M = \{\beta_{ij} \in B \,:\, \operatorname{supp} \beta_{ij} \cap \mathcal{M}^\circ \neq \emptyset\},$$

where $\mathcal{M}^\circ$ denotes the interior of $\mathcal{M}$, see Fig. 7 for an example. These basis functions generate a space which we denote by

$$\mathbb{V}_M = \operatorname{span} B_M|_M. \tag{7}$$

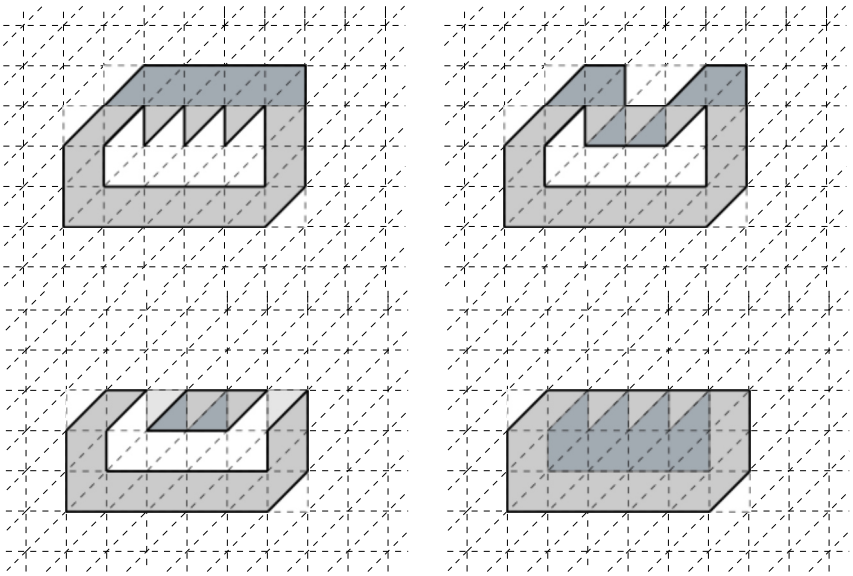

FIGURE 6. Examples of sets M satisfying Condition 5.

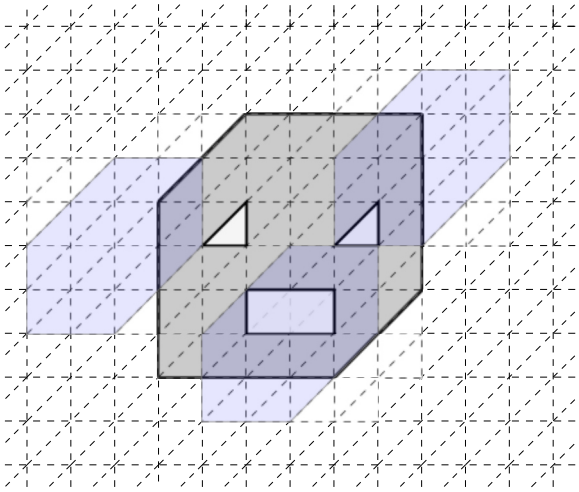

FIGURE 7. Supports (shown in blue) of some of the translates of the box splines $\mathcal{B}$ that act on a domain $\mathcal{M}$ (grey).

In particular, when M contains just the single cell $\triangle$ then

$$\mathbb{V}_M = \mathbb{V}_{\{\triangle\}} = \mathbb{V}_{\triangle} = \hat{\mathcal{P}}|_{\triangle}$$

as we defined in Remark 1.

Definition 7. For a finite set of triangles $M \subset G$ let

$$\mathbb{D}_M = \{(f|_\triangle)_{\triangle\in M} \,:\, f|_\triangle \in \mathcal{P}|_\triangle\}\ ,\ \text{and}\ \hat{\mathbb{D}}_M = \left\{(f|_\triangle)_{\triangle\in M} \,:\, f|_\triangle \in \hat{\mathcal{P}}|_\triangle\right\},$$

be the spaces of **disconnected quartics** and **disconnected special quartics on** M, respectively.

For $M = \{\triangle\}$, the spaces $\mathbb{D}_M$ and $\hat{\mathbb{D}}_M$ coincide with $\mathcal{P}|_\triangle$ and with $\hat{\mathcal{P}}|_\triangle$. It is obvious that $\hat{\mathbb{D}}_M \subset \mathbb{D}_M$ for any choice of M.

Given a disconnected special quartic $\mathbf{f} = (f|_\triangle)_{\triangle\in M} \in \hat{\mathbb{D}}_M$, we have a local representation

$$f|_\triangle(\mathbf{x}) = \sum_{\beta\in B} \lambda_\triangle^\beta(f|_\triangle)\beta|_\triangle(\mathbf{x}),\ \mathbf{x}\in\triangle,$$

in terms of the restriction of the box splines, for any $\triangle \in M$. However, this representation is generally not available for general disconnected quartics.

Definition 8. For a finite set of triangles M, and the corresponding disconnected space $\hat{\mathbb{D}}_M$, the **special spline space** on M is defined by

$$\hat{\mathbb{S}}_M = \{\mathbf{f} \in \hat{\mathbb{D}}_M \,:\, \forall\triangle,\triangle' \in M,\ f|_\triangle \sim f|_{\triangle'}\}$$

where the relation $\sim$ is defined in Definition 2.

As we shall see later, the special spline space $\hat{\mathbb{S}}_M$ can be generated by box splines with support on $\mathcal{M}$, but one may need to make several copies of some of these box splines, as shown in the following Example.

Example 9. The domain in Fig. 8 consists of two $\star$-disconnected triangles. The space of disconnected special quartics consists of pairs of special polynomials, where the first and the second entry of each pair is associated with the first and the second triangle. Since the two triangles are disconnected, the special spline space is equal to the space of disconnected special quartics. Consequently, it has dimension 24 and is therefore not spanned by the 18 box splines whose support intersects this domain. ◇

Any $\mathbf{f} = (f|_\triangle)_{\triangle\in M} \in \hat{\mathbb{S}}_M$ defines a unique continuous function $\tilde{\mathbf{f}}$ on $\mathcal{M}$, with the property that

$$\tilde{\mathbf{f}}(\mathbf{x}) = f|_\triangle(\mathbf{x}) \quad \text{for every } \mathbf{x}\in\triangle, \text{and } \triangle \in M.$$

Using the representation of $\mathbf{f}$ as an element in $\hat{\mathbb{D}}_M$ and the characteristic functions $\chi_\triangle$ of the cells $\triangle \in M$, we may also write $\tilde{\mathbf{f}}$ as

$$\tilde{\mathbf{f}}(\mathbf{x}) = \sum_{\triangle\in M}\sum_{\beta\in B_\triangle} \lambda_\triangle^\beta(f|_\triangle)\beta(\mathbf{x})\chi_c^\star(\mathbf{x}), \quad \text{for } \mathbf{x}\in\mathcal{M}$$

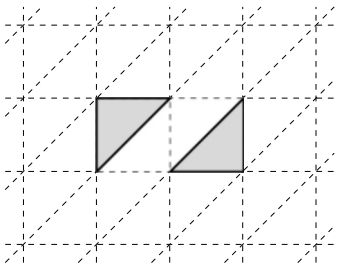

FIGURE 8. Domain with $\mathbb{V}_M \subsetneq \hat{\mathbb{S}}_M$

where

$$\chi^{\star}_{\triangle}(\mathbf{x}) = \begin{cases} \dfrac{\chi^{\star}_{\triangle}(\mathbf{x})}{\sum_{c\in M}\chi_c(\mathbf{x})}, & \text{if } x \in \triangle \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

We will use the same notation $\mathbf{f}$ for the elements of $\hat{\mathbb{S}}_M$ which are $|M|$-tuples of polynomials (where $|M|$ is the number of triangles in M), and the actual spline functions $\tilde{\mathbf{f}}$. Consequently, we identify the special spline space $\hat{\mathbb{S}}_M$ with the space $S^2(M, \hat{\mathcal{P}})$, i.e.,

$$\hat{\mathbb{S}}_M = S^2(M, \hat{\mathcal{P}}). \tag{9}$$

Clearly, this space is a subset of the full quartic spline space $S^2(M, \mathcal{P})$.

Definition 10. For a spline $\beta \in B_M$, the coefficient graph Γ_β associated to β is defined as follows:

- The vertices of the graph Γ_β are the cells $\triangle \in M$ such that $\triangle \subseteq \operatorname{supp}\beta$.
- Two vertices of Γ_β are connected by an edge if the corresponding cells $\triangle, \triangle'$ have a common edge and $\beta|_{\bar{\triangle}\cap\bar{\triangle}'} \neq 0|_{\bar{\triangle}\cap\bar{\triangle}'}$.

We will write $\triangle \in \Gamma_\beta$ to indicate that $\triangle$ corresponds to a vertex of Γ_β.

Example 11. Let us consider the domain in Fig. 9, and the box splines β_i $(i = 1, 2, 3)$. The corresponding coefficient graphs Γ_{β_i} are given in Fig. 10. ◇

Proposition 12. An element $\mathbf{f} \in \mathbb{D}_M$ is in $\hat{\mathbb{S}}_M$ if and only if the coefficients satisfy $\lambda^{\beta}_{\triangle}(f|_{\triangle}) = \lambda^{\beta}_{\triangle'}(f|_{\triangle'})$, for all $\beta \in B_M$, and all pair of cells $\triangle, \triangle'$ belonging to the same component of Γ_β.

Proof. Suppose $\mathbf{f} \in \hat{\mathbb{S}}_M$, and β in $\mathcal{B}$. If $\triangle = \triangle_0$ and $\triangle' = \triangle_{d+1}$ are two cells in M corresponding to vertices in the same component of

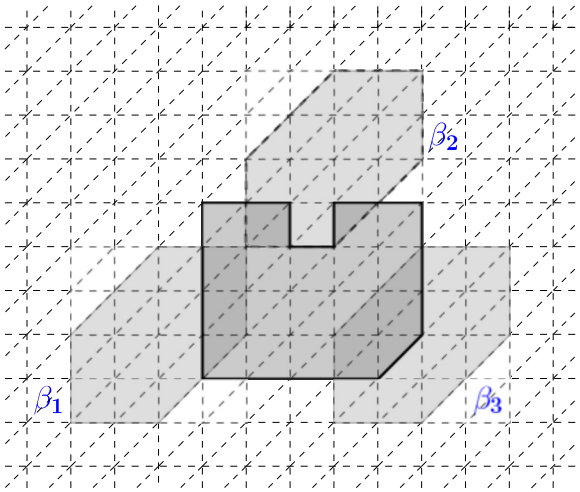

FIGURE 9. Triangular domain, and three examples of box splines β_i.

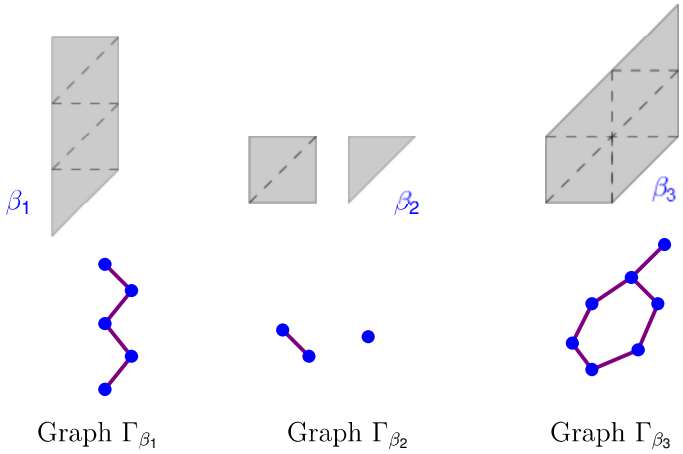

FIGURE 10. Coefficient graphs associated to β_i, $i = 1, 2, 3$ from Fig. 9.

Γ_β, then there is a chain of vertices $v_1, \ldots, v_d$ in Γ_β corresponding to cells $\triangle_1, \ldots, \triangle_d$ in M, such that $\triangle_i$ and $\triangle_{i+1}$ intersect in an edge, for $i = 0, \ldots, d$. By Lemma 3, $\lambda^\beta_{\triangle_i}(f|_{\triangle_i}) = \lambda^\beta_{\triangle_{i+1}}(f|_{\triangle_{i+1}})$, and since this is valid for every $0 \leq i \leq d$, then the same follows for $\triangle$ and $\triangle'$.

Conversely, from a similar argument as before, if for any pair of triangles $\triangle$ and $\triangle'$ in M having an edge e in common we have that $\lambda^\beta_\triangle(f|_\triangle) = \lambda^\beta_{\triangle'}(f|_{\triangle'})$ for every basis function $\beta \in B$ such that $\beta|_e \neq 0$,

then by Lemma 3 every linear combination of the basis functions β is in $\hat{\mathbb{S}}_M$. □

3.2. **Box spline bases on triangular domains.**

Definition 13. For every $\beta \in B$ and every connected component Φ of Γ_β we define the function

$$\beta_\Phi(\mathbf{x}) = \sum_{c \in \Phi} \beta(\mathbf{x}) \chi_c^\star(\mathbf{x}),$$

where $\chi_c^\star(\mathbf{x})$ is defined as in (8). The set of these functions, for the different connected components of the graph Γ_β, is denoted by Λ i.e.,

$$\Lambda = \bigcup_{\beta \in B} \{\beta_\Phi |\ \Phi \text{ is a connected component of } \Gamma_\beta\}.$$

Theorem 14. *The set Λ, when restricted to $\mathcal{M}$, forms a locally linearly independent basis for $\hat{\mathbb{S}}_M$.*

The proof is analogous to that of Theorem 2.12 in [13].

Corollary 15. *If the intersection of the support of each β with the multi-cell domain $\mathcal{M}$ is $\star$-connected, then the functions in B_M, when restricted to $\mathcal{M}$, form a basis of $\hat{\mathbb{S}}_M$.*

Proof. If this condition is satisfied, then for each $\beta \in B_M$ the coefficient graph Γ_β has either one component or it is empty, and the result follows from the theorem. □

Example 16. The graph Γ_β, associated to every β with no empty intersection with the interior of $\mathcal{M}$ in Fig. 7, has only one component. From the previous corollary, it follows that the functions in B_M, restricted to M, form a basis for the special spline space $\hat{\mathbb{S}}_M$ on the domain $\mathcal{M}$. ◇

4. Characterization of admissible domains

Definition 17. A domain $\mathcal{M} = \bigcup M$ is said to be **admissible**, if the intersection of the support of any box-spline with $\mathcal{M}$ is $\star$-connected.

The following result is then obvious from Corollary 15:

Corollary 18. *For any admissible domain $\mathcal{M}$, the functions in B_M when restricted to $\mathcal{M}$ form a basis of $\hat{\mathbb{S}}_M$.*

A subset of admissible domains can be characterized by the offsets of their boundaries.

Definition 19. We define the offset curve of a multi-cell domain M as follows: consider any cell (triangle) in $G \setminus M$. If the boundary of this triangle shares a vertex with $\mathcal{M}$, but both incident edges are not part of the boundary of $\mathcal{M}$, then the opposite edge is added to the offset curve. We say that a domain $\mathcal{M}$ satisfies the offset condition (OC), if its offset is a simple closed curve or a collection of simple closed curves.

Proposition 20. If a domain satisfies the offset condition, then it is also admissible.

Proof. The proof follows from a careful case-by-case analysis. □

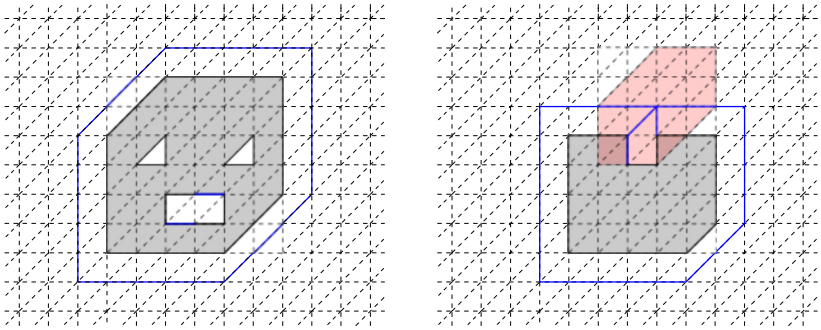

FIGURE 11. Offset curves.

Remark 21. For the domain on the left in Fig. 11 the box splines in B_M form a basis for $\hat{\mathbb{S}}_M$. In this situation, when the holes in the domain are "sufficiently small", they do not split the domain of any basis function $\beta \in \mathcal{B}_M$ and the result follows by Corollary 15. Consequently, the offset condition is sufficient, but not necessary, for admissibility. ⋄

5. Hierarchical Box splines

In this section we define hierarchical box spline and we prove the completeness of the hierarchical construction, i.e. the fact that the hierarchical basis spans locally $\mathcal{P}^\star$ and has C^2 continuity along the edges of the hierarchical mesh.

5.1. **Hierarchies of box spline spaces.** Let $N \geq 0$ be an integer, and consider the grids

$$G^\ell \quad , \quad \ell = 0, \ldots, N-1$$

such that $G^{\ell+1}$ is obtained from G^ℓ by one global, uniform dyadic refinement step. By dyadic refinement we mean that every triangle-edge of G^ℓ is divided into two edges, and the new edges are added, thus every triangle is split into four smaller ones, see Fig. 12.

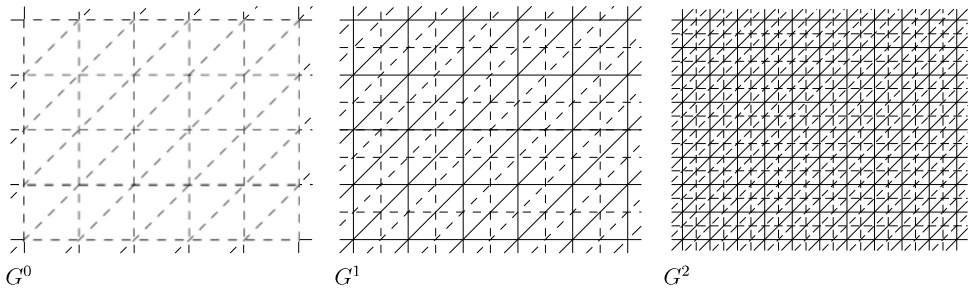

FIGURE 12. Hierarchical grids.

The corresponding spline functions B^ℓ define the spline spaces $\mathbb{V}^\ell = \operatorname{span} B^\ell$. Since the grids are nested, the respective spaces are also nested, i.e.

$$\mathbb{V}^\ell \subseteq \mathbb{V}^{\ell+1} \quad , \quad \ell = 0, \dots, N-1 \ .$$

We are going to construct a hierarchical spline space over these nested grids. To this end, the following definition is convenient:

Definition 22. For a real subdomain $\mathcal{M} \in \mathbb{R}^2$, we define

$$T^\ell(\mathcal{M}) = \{\triangle \in G^\ell \ : \ \triangle \subseteq \mathcal{M}\},$$

and for a set of triangles $M^\ell \subset G^\ell$ the union operation

$$\boldsymbol{\vdash\!\!\!\!\!\!\dashv}_{M^\ell} = \boldsymbol{\vdash\!\!\!\!\!\!\dashv}_{\triangle \in M^\ell} \triangle \subset \mathbb{R}^2 \ .$$

Using this notation, we can distinguish between a set of triangles, and the real sub-domain occupied by these triangles.

Let Ω be a domain of $\mathbb{R}^2$ aligned with level $N-1$. That is, $\partial\Omega$ is a union of edges from the grid G^{N-1}.

Now, let us start with an inversed nested sequence of domains $\mathcal{M}^\ell$,

$$\mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots \subseteq \mathcal{M}^{N-1} \ .$$

with $\partial\mathcal{M}^\ell \subset G^\ell$, for $\ell = 0, \dots, N-1$, and let $M^\ell = T^\ell(\mathcal{M})$, see Figures 13 and 14. We denote $\Omega = \mathcal{M}^{N-1}$ (since it needs to be aligned with level $N-1$). These sets are called rings in [5].

We assume that for each ℓ, the multi-cell domains M^ℓ satisfy Condition 5. i.e., all the components of M^ℓ are $\star$-connected. Then, the box-splines in B^ℓ whose support has a no empty intersection with $\mathcal{M}^\ell$ form a (locally linearly independent) basis of $\hat{\mathbb{S}}_{M^\ell}$, according to Theorem 14.

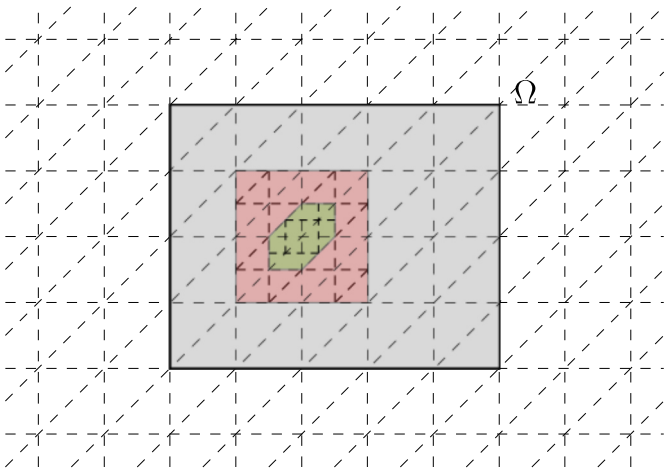

FIGURE 13. Partition of a domain Ω as a hierarchical mesh, containing cells from the grids G^0, G^1 and G^2 in Fig. 12.

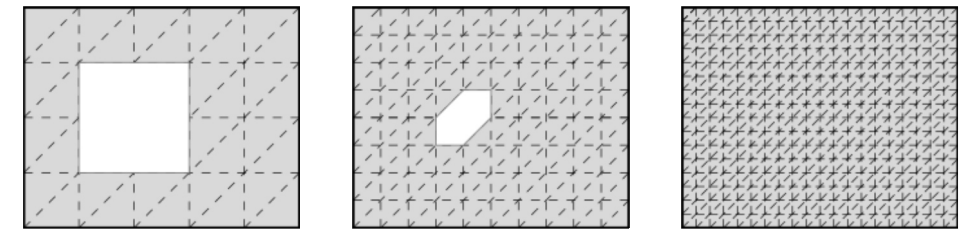

FIGURE 14. Multi-cell domains in the grids G^i, for $i = 0, 1, 2$ from Fig. 12, respectively.

The difference of two successive multi-cell domains $M^{\ell-1}$ and M^ℓ is called refinement area of level ℓ, formally

$$D^\ell = M^\ell \setminus T^\ell(\mathcal{M}^{\ell-1}),$$

and the corresponding real domains

$$\mathcal{D}^\ell = \overline{\bigcup D^\ell} = \overline{\mathcal{M}^\ell \setminus \mathcal{M}^{\ell-1}}.$$

Consequently, these sets define a partition of Ω as a hierarchical mesh:

$$\Omega = \bigcup_{i=0}^{N-1} \mathcal{D}^i = \overline{\bigcup_{\ell=0}^{N-1} \bigcup D^\ell}$$

so that Ω is regarded as a disjoint partition of cells of different levels, containing exactly the cells D^ℓ from grid G^ℓ, and M^ℓ is written as

$$M^\ell = \bigcup_{i=0}^{\ell} T^\ell(D^i) \ . \tag{10}$$

In particular we have $D^0 = M^0$, see Fig. 13.

We shall introduce a hierarchy of spline spaces defined on these domains. For each level ℓ, we consider the spline space $\hat{\mathbb{S}}_{M^\ell}$, so that

$$\mathbb{V}^\ell_{M^\ell} \subseteq \hat{\mathbb{S}}^\ell_{M^\ell} \ .$$

The spaces $\mathbb{V}^\ell_{M^\ell}$ and $\hat{\mathbb{S}}^\ell_{M^\ell}$ are defined as in Eq. (7) and Definition 8, but using the spline space and the grid of level ℓ.

Given a multi-cell domain $D^\ell \subset G^\ell$, we have again the inclusion

$$\mathbb{V}^\ell|_{\mathcal{D}^\ell} \subsetneq S^2_4(D_\ell) \quad , \quad \ell = 0, \dots, N-1.$$

We are now able to define the spline space we are interested in

Definition 23. The hierarchical special spline space $\mathbb{H}$ is defined as

$$\mathbb{H} = S^2\left(\bigcup_{\ell=0}^{N-1} D^\ell, \hat{\mathcal{P}}\right)$$

Each cell of D^ℓ is also contained in M^ℓ, and on the other hand, each cell of M^ℓ is contained in a cell of some D^k for $k \leq \ell$, cf. (10). Consequently, we can characterize the space $\mathbb{H}$ equivalently as

$$\mathbb{H} = \left\{ h : \Omega \to \mathbb{R} \mid \forall \ell : h|_{\mathcal{M}^\ell} \in \hat{\mathbb{S}}^\ell_{M^\ell} \right\},$$

since, by (9), we have that

$$\hat{\mathbb{S}}^\ell_{M^\ell} = S^2(M^\ell, \hat{\mathcal{P}}). \tag{11}$$

Note that this is more general than requiring $h|_{\mathcal{M}^\ell} \in \mathbb{V}^\ell|_{M^\ell}$. In fact, the space $\mathbb{H}$ contains any C^2-smooth function with the property that its restriction to any cell in the hierarchical grid is a special quartic polynomial. This is expressed by requiring that the restriction of such a function to the multi-cell domain $\mathcal{M}^\ell$ belongs to the special spline space on M^ℓ. In contrast, the space $\mathbb{V}^\ell_{M^\ell}$ contains only those functions that can be represented as linear combinations of restrictions of box splines to $\mathcal{M}^\ell$.

5.2. **The box spline basis of hierarchical special splines.** Finally we formulate the main result of our paper.

Proposition 24. If each multi-cell domain M^ℓ is admissible for every $\ell = 0, \dots, N-1$, then the functions in

$$\mathcal{K} = \bigcup_{\ell=0}^{N-1} \mathcal{K}^\ell$$

forms a basis for the space of hierarchical box splines $\mathbb{H}$, where

$$\mathcal{K}^\ell = \left\{ \beta^\ell \in B^\ell_{M^\ell} \,|\, \operatorname{supp} \beta \cap \mathcal{M}^{\ell-1} = \emptyset \right\}.$$

Proof. The proof follows standard arguments already presented in [5, 13], for the tensor B-spline basis. First we verify that $\mathcal{K}$ is linearly independent. This is deduced from the fact that every $\mathcal{K}^\ell$ is locally linearly independent, from the properties of quartic box splines.

Let $h \in \mathbb{H}$, by the definition of the hierarchical space, there exists $h^\ell = h|_{\mathcal{M}^\ell} \in \hat{\mathbb{S}}^\ell_{M^\ell}$, which implies $h^\ell \in \operatorname{span} \mathcal{K}^\ell$, by Corollary 15. We obtain

$$h^\ell|_{\mathcal{M}^\ell} = h - \sum_{i=0}^{\ell-1} h^i$$

leading to $h = h^\ell + r^\ell$, where the residual r^ℓ has the property $r^\ell|_{\mathcal{M}^\ell} = 0$.

Considering this argument for all $\ell = 1, \dots, N-1$, we arrive at

$$h = h^0 + r^0 = h^0 + h^1 + r^1 = \cdots = \sum_{i=0}^{N-1} h^\ell + r,$$

where the residual function is

$$r = \sum_{\ell=0}^{N-1} r^\ell.$$

Since the $\mathcal{M}^\ell$ are inversly nested,

$$r = r|_\Omega = r^{N-1}|_{\mathcal{M}^{N-1}} = 0,$$

so that $h \in \mathcal{K}$, as needed. □

6. Conclusion

We extended the discussion of the completeness of hierarchical spline spaces from [13] to the case of hierarchies of bivariate quartic C^2-smooth box splines on type-I tria tringulations. There two main differences to the original approach, which was formulated for tensor-product splines.

First, since box splines do not span the whole space of quartic polynomials, a special polynomial subspace – the special quartics – had to

be introduced. In some sense this is even similar to the tensor-product case, where the B-splines span a tensor-product space instead of the the space of polynomials of a given degree.

Second, the constraints on the domains are entirely different, due to the differences in the characterization of contacts between polynomial pieces. For bivariate tensor-product splines, both edge-edge and vertex-vertex contacts could be characterized easily by the equality of spline coefficients. In the present case, however, this was possibly solely for edge-edge contacts. Consequently, the completeness of hierarchical splines requires more severe restrictions to the hierarchical grid.

The hierarchical box-spline basis does not form a partition of unity. Similar to the approach presented in [6], this property can be recovered with the help of a suitable truncation procedure. Also, [13] described a decoupling procedure that allows to relax the assumptions regarding the hierarchical grid. This approach can be extended to the box spline case as well. Finally it is also possible to combine truncation and decoupling as in [12].

Appendix I

We show that the space $\mathbb{V}_{\triangle}$ defined in Eq. (3) is the restriction of a global space to the triangle, as pointed out in Remark 1.

The following proof is not restricted to quartic box-splines.

Proposition 25. *Consider a global polynomial $f \in \mathcal{P}$ defined in $\mathbb{R}^2$. If $f|_{\triangle} \in \mathbb{V}_{\triangle}$ for some $\triangle \in G^\ell$, then $f|_{\triangle'} \in \mathbb{V}_{\triangle'}$ for any other cell $\triangle'$ in the grid G^ℓ.*

Proof. Suppose that $\triangle, \triangle' \in G^\ell$ and that both $\triangle$ and $\triangle'$ are contained in a bigger triangle $\tilde{\triangle}$ of a grid which we denote as G^0. We denote by $\mathbb{V}^0$ the span of the box splines (associated to the grid G^0) restricted to $\tilde{\triangle}$. In this way, $\triangle$ and $\triangle'$ are obtained from $\tilde{\triangle}$ after ℓ successive refinement steps. Consequently, $\mathbb{V}_{\triangle} = \mathbb{V}^\ell_{\triangle}$, and $\mathbb{V}_{\triangle'} = \mathbb{V}^\ell_{\triangle'}$

Let $\mathbb{V}^0|_{\triangle}$ be the space of polynomials in $\mathbb{V}^0$ restricted to the cell $\triangle$. It is clear that $\dim \mathbb{V}^0 = \dim \mathbb{V}^\ell_{\triangle}$, by definition of the box spline space. On the other hand, notice that by the refinement procedure $\mathbb{V}^0|_{\triangle} \subseteq \mathbb{V}^\ell_{\triangle}$. Hence, $\mathbb{V}^0|_{\triangle} = \mathbb{V}^\ell_{\triangle}$.

Similarly, taking the restriction $\mathbb{V}^0|_{\triangle'}$, we can see that locally $\mathbb{V}^0|_{\triangle'} = \mathbb{V}^\ell_{\triangle'}$. Therefore, $f|_{\triangle'} \in \mathbb{V}_{\triangle'}$. □

We may therefore define $\hat{\mathcal{P}}$ to be this global polynomial space. In the special case of C^2 quartic box splines, the above result may be seen directly by using the representation of the polynomial pieces in the monomial basis.

More precisely, from the local coefficients of the quartic box splines, we can find a basis for $\hat{\mathcal{P}}$ expressed in the monomial form as:

$$\hat{\mathcal{P}} = \text{span}\,(\mathcal{P}_3 \cup \{x^4 - 2x^3y, y^4 - 2xy^3\}).$$

Remark 26. We can use this basis of $\hat{\mathcal{P}}$ to make a proof of Proposition 25 for the particular case of C^2 quartic box splines as follows.

With the notation as in Proposition 25, by a change of coordinates we can assume that $\triangle$ has a vertex at $(0,0)$. Since $f \in \mathbb{V}_{\triangle}$, then

$$f(x,y) = \tilde{f}(x,y) + c_1(x^4 - 2x^3y) + c_2(y^4 - 2xy^3)\,,$$

where $\tilde{f} \in \mathcal{P}_3$ and c_1, c_2 are the coefficients of the degree four part. Since $\triangle'$ is a translation of $\triangle$, then by the change of coordinates $x = x' + a$, $y = y' + b$ where (a,b) is a vertex of $\triangle'$, it is enough to show that f in this new coordinate system belongs to $\mathbb{V}_{\triangle'}$. Indeed,

$$f(x',y') = f(x+a, y+b) = \tilde{f}'(x',y') + c_1(x'^4 - 2x'^3y') + c_2(y'^4 - 2x'y'^3)\,,$$

where $\tilde{f}'$ is a cubic polynomial. ◇

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