

## Inf-sup stability of isogeometric Taylor-Hood and SubGrid methods for the Stokes problem with hierarchical splines

ANDREA BRESSAN <sup>†</sup> AND BERT JÜTTLER <sup>‡</sup>

*Institute for Applied Geometry, Johannes Kepler Universität, Linz, Austria*

[Received on ???; revised on ???]

In this paper we extend the proof of the inf-sup-stability of isogeometric Taylor-Hood (TH) and Sub-Grid (SG) elements from tensor-product to hierarchical spline spaces.

*Keywords:* Stokes; hierarchical splines; IGA; ins-sup stability

### 1. Introduction

The *Taylor-Hood* (TH) element is a well known mixed discretization both of incompressible elasticity and incompressible fluid dynamic. The generalization of the TH method to *isogeometric analysis* (IGA) was initially studied by Bazilevs *et al.* (2006) for  $C^0$  discretizations (i.e. isoparametric FEM) and later extended to smoother function spaces:  $C^1$  by Bressan (2011) and  $C^k$  discretizations,  $k \geq 0$  by Bressan & Sangalli (2013). The latter contains general conditions for stability and two families of inf-sup-stable pairs: the isogeometric TH, and the isogeometric *Sub-Grid* (SG) element. In the SG method, stability is obtained by refining the velocity grid, the same idea was independently proposed by Rüberg & Cirak (2012). The cited papers deal only with tensor-product spline spaces. In this paper we recall the theory and extend the stability result to adaptive methods based on hierarchical splines. In our knowledge this is the first extension of TH and SG methods to non-tensor-product splines. Note that there are also div-conforming elements for the Stokes problem, which we do not consider in this paper. Div-conforming methods with tensor-product spline were studied by Buffa *et al.* (2011); Evans & Hughes (2013a,b). Then they were extended to T-splines (Buffa *et al.*, 2014), to LR-splines (Johannessen *et al.*, 2015) and hierarchical splines (Evans *et al.*, 2015).

In order to make the paper self-contained, we recall the structure of the proof for tensor-product splines in Section 2. In Section 3 we quickly introduce hierarchical splines, we prove the properties required by the results recalled in Section 2 and conclude that a discretization based on hierarchical splines is stable under mild assumptions on the mesh structure (Theorem 3.1). In Section 4 we report numerically computed inf-sup constants for meshes that conform and do not conform to the hypothesis of the stability result. In Section 5 we report the results of the application of a simple adaptive method based on hierarchical splines to a problem having corner singularities.

### 2. Preliminaries

This section is composed of three independent parts: the description of the inf-sup theory in the context of the Stokes problem, a quick description of isogeometric analysis, and a summary of the techniques

<sup>†</sup>Corresponding author. Email: andrea.bressan@jku.at

<sup>‡</sup>Email: bert.juettler@jku.at

used for assessing stability of the TH element in FEM and the TH and SG method in IGA.

### 2.1 Stokes problem and inf-sup stability

The Stokes system of equations is a linearized model of the stationary flow of a nearly incompressible fluid under the assumption of negligible inertia. The unknowns are the velocity of the fluid  $\mathbf{u}$  and the pressure  $p$  in the domain  $\Omega \subset \mathbb{R}^n$ . The data are the fluid viscosity  $\nu$ , the volume forces  $\mathbf{f}$ , the boundary forces  $\mathbf{u}_N$  on  $\Gamma_N \subseteq \partial\Omega$  and the velocity values  $\mathbf{u}_D$  on  $\Gamma_D = \partial\Omega \setminus \Gamma_N$ . We also include a source term  $g$  that is usually absent in physical models. The Stokes system is

$$\begin{cases} -\nu\Delta\mathbf{u} + \text{grad } p = \mathbf{f} \\ -\text{div } \mathbf{u} = g \\ \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D \\ \text{grad } u \cdot \mathbf{n} - np = \mathbf{u}_N \text{ on } \Gamma_N. \end{cases} \quad (2.1)$$

Note that if  $\Gamma_N = \emptyset$ , then  $p$  is determined solely up to a constant because only its gradient  $\text{grad } p$  appears in (2.1). Similarly if  $\Gamma_D = \emptyset$  then  $\mathbf{u}$  is determined only up to a constant vector. These degenerate cases can be handled by constraining the mean value of  $p$  or  $\mathbf{u}$  respectively.

In order to simplify the notation we fix  $\nu = 1$  and write the formulas for the case  $\mathbf{u}_D = 0$  and  $\mathbf{u}_N = 0$ . The general case can be reduced to this by the following strategies:

- If  $\mathbf{u}_D \neq 0$ , then it is possible to choose  $\mathbf{z} : \Omega \rightarrow \mathbb{R}^n$  such that  $\mathbf{z}|_{\Gamma_D} = \mathbf{u}_D$  and solve for  $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{z}$  with data  $\tilde{\mathbf{f}} = \mathbf{f} - \Delta\mathbf{z}$  and  $\tilde{g} = g - \text{div } \mathbf{z}$ ;
- If  $\mathbf{u}_N \neq 0$ , then the boundary term can be included in the force term in the weak formulation of the problem.

The Stokes problem can be written as a mixed formulation by setting

$$a(\mathbf{w}, \mathbf{u}) = \int_{\Omega} \text{grad } \mathbf{w} : \text{grad } \mathbf{u}, \quad b(\mathbf{w}, q) = - \int_{\Omega} \text{div } \mathbf{w} q.$$

Then (2.1) becomes: find  $(\mathbf{u}, p) \in V \times P$  such that  $\forall (\mathbf{w}, q) \in V \times P$

$$\begin{cases} a(\mathbf{u}, \mathbf{w}) + b(\mathbf{w}, p) = f(\mathbf{w}) \\ b(\mathbf{u}, q) = g(q), \end{cases} \quad (2.2)$$

where  $V$  and  $P$  are function spaces,  $a : V \times V \rightarrow \mathbb{R}$  and  $b : V \times P \rightarrow \mathbb{R}$  are continuous bilinear forms and  $f \in V^*$ ,  $g \in P^*$  represent both the source and the boundary data. The kernel of  $B : V \rightarrow P^*$  defined by  $B(\mathbf{w})(q) = b(\mathbf{w}, q)$  plays an important role in the analysis of problem (2.2) as described in Theorem 1.1 of Brezzi & Fortin (1991). A specialization for symmetric  $a$  and  $\ker B = \{0\}$  is reported below:

**THEOREM 2.1** There exists a unique solution  $(\mathbf{u}, p)$  of (2.2) if

- $\exists \alpha > 0 : \forall \mathbf{w} \in \ker B, a(\mathbf{w}, \mathbf{w}) \geq \alpha \|\mathbf{w}\|_V^2$
- $\exists \beta > 0 : \forall q \in P, \exists \mathbf{w} \in V : b(\mathbf{w}, q) \geq \beta \|\mathbf{w}\|_V \|q\|_P.$

Moreover the solution satisfies:

$$\|\mathbf{u}\|_V \leq \alpha^{-1} \|f\|_{V^*} + \beta^{-1} (1 + \alpha^{-1} \|a\|) \|g\|_{P^*} \quad (2.3)$$

$$\|p\|_P \leq \beta^{-1} (1 + \alpha^{-1} \|a\|) \|f\|_{V^*} + \beta^{-2} \|a\| (1 + \alpha^{-1} \|a\|) \|g\|_{P^*}. \quad (2.4)$$

Condition (i) is called *coercivity on the kernel* and condition (ii) is called *inf-sup condition*.

A *conformal* discretization of (2.2) is a restriction to a finite dimensional subspace: find  $(\mathbf{u}_h, p_h) \in V_h \times P_h \subseteq V \times P$  such that  $\forall (\mathbf{w}, q) \in V_h \times P_h$

$$\begin{cases} a(\mathbf{u}_h, \mathbf{w}) + b(\mathbf{w}, p_h) = f(\mathbf{w}) \\ b(\mathbf{u}_h, q) = g(q) \end{cases} \quad (2.5)$$

Theorem 2.1 applies to (2.5), but the validity of conditions (i) and (ii) for the pair  $V, P$  does not imply their validity for the pair  $V_h, P_h$ . The main result concerning the discretization error is Theorem 2.1 in Brezzi & Fortin (1991) which in our case can be written as:

**THEOREM 2.2** The difference between the solutions  $(\mathbf{u}, p)$  of (2.2) and  $(\mathbf{u}_h, p_h)$  of (2.5) is bounded by

$$\|\mathbf{u} - \mathbf{u}_h\|_V \leq \left(1 + \frac{\|a\|}{\alpha_h}\right) \left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{\mathbf{w} \in V_h} \|\mathbf{u} - \mathbf{w}\|_V + \frac{\|b\|}{\alpha_h} \inf_{q \in P_h} \|p - q\|_P; \quad (2.6)$$

$$\|p - p_h\|_P \leq \left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{q \in P_h} \|p - q\|_P + \frac{\|a\|}{\beta_h} \|\mathbf{u} - \mathbf{u}_h\|_V \quad (2.7)$$

provided that conditions (i) and (ii) hold with constants  $\alpha_h, \beta_h$  for the pair  $V_h, P_h$ .

Appropriate spaces for the Stokes problem are

$$V = \begin{cases} \{\mathbf{w} \in H^1(\Omega)^n : \mathbf{w}|_{\Gamma_D} = 0\} & \Gamma_D \text{ has positive } n-1 \text{ dimensional volume} \\ \{\mathbf{w} \in H^1(\Omega)^n : \int_{\Omega} \mathbf{w} = 0\} & \text{otherwise} \end{cases} \quad (2.8)$$

with norm  $\|\mathbf{w}\|_V = |\mathbf{w}|_{H^1(\Omega)^n} = a(\mathbf{w}, \mathbf{w})^{\frac{1}{2}}$  and

$$P = \begin{cases} L^2(\Omega) & \Gamma_N \text{ has positive } n-1 \text{ dimensional volume} \\ \{q \in L^2(\Omega) : \int_{\Omega} q = 0\} & \text{otherwise} \end{cases} \quad (2.9)$$

with norm  $\|q\|_P = \|q\|_{L^2}$ . Then  $\alpha = \alpha_h = \|a\| = 1$ ,  $\|b\| = 1$  and (i) is fulfilled for all pairs  $V_h, P_h$ .

Condition (ii) for  $V, P$  depends on the shape of the domain. A positive  $\beta$  as in (ii) exists if  $\Omega$  is a John domain (Acosta *et al.*, 2006; Costabel *et al.*, 2015) and thus, in particular, if  $\Omega$  is a Lipschitz domain. The discrete inf-sup condition depends on the choice of  $V_h, P_h$ . When it holds, Theorem 2.2 provides the following error estimates:

$$\|\mathbf{u} - \mathbf{u}_h\|_V \leq 2 \frac{1 + \beta_h}{\beta_h} \inf_{\mathbf{w} \in V_h} \|\mathbf{u} - \mathbf{w}\|_V + \inf_{q \in P_h} \|p - q\|_P; \quad (2.10)$$

$$\|p - p_h\|_P \leq 2 \frac{1 + \beta_h}{\beta_h^2} \inf_{\mathbf{w} \in V_h} \|\mathbf{u} - \mathbf{w}\|_V + \frac{2 + \beta_h}{\beta_h} \inf_{q \in P_h} \|p - q\|_P. \quad (2.11)$$

The above inequalities show the competing behaviour of the approximation properties of  $P_h$  and the inf-sup condition. On one hand, choosing a larger space  $P_h$  decreases the approximation error  $\inf_{q \in P_h} \|p - q\|_P$ , on the other hand, it can simultaneously decrease the inf-sup constant  $\beta_h$ . Vice-versa choosing a smaller space  $P_h$  provides weaker approximation properties and can increase the inf-sup constant. These inequalities also show that the existence of a common lower bound  $\beta_0 \leq \beta_h$  for all the pairs  $V_h, P_h$  defined by a method implies the quasi-optimality of the error for the method.

## 2.2 Isogeometric Analysis

Isogeometric Analysis (IGA) is a framework that tries to streamline the design process of technological products. It was introduced by Hughes *et al.* (2005), it is described in the monograph of Cottrell *et al.* (2009) and analyzed mathematically in Beirão da Veiga *et al.* (2014). It aims at integrating the Computer Aided Design (CAD) and Computer Aided Engineering (CAE) by sharing the same geometry representation techniques. In the traditional approach the geometry provided by CAD must be converted to a mesh on which the simulation can be performed. Then, in case of geometry optimization, the result is re-imported into the CAD system. Experience suggests that the conversions of the geometry representation is responsible for significant percentage of the product development time. IGA tries to optimize the complexity of the whole process, by reducing the conversion cost.

IGA uses common industrial technology: in particular NURBS functions and parametrizations. The physical domain  $\Omega$  is assumed to be the image of a NURBS map  $G : \hat{\Omega} = [0, 1]^n \rightarrow \mathbb{R}^k$ . This is the most common method to represent a domain in CAD. Then the isoparametric approach is applied: the discretization is constructed on the parametric domain  $\hat{\Omega}$  and pushed to the physical domain  $\Omega$ . Following the IGA convention, quantities related to the parametric domain are marked with a hat:  $\hat{\cdot}$ .

The discretization spaces in IGA are the push-forward of spline (or NURBS) spaces defined in the parametric domain

$$V_h = \left\{ \mathbf{w} = \hat{\mathbf{w}} \circ G^{-1} : \hat{\mathbf{w}} \in \hat{V}_h \right\} \cap V \quad (2.12)$$

$$P_h = \left\{ q = \hat{q} \circ G^{-1} : \hat{q} \in \hat{P}_h \right\} \cap P. \quad (2.13)$$

The approximation error is studied in the parametric domain and then the result is transferred on the physical domain (Beirão da Veiga *et al.*, 2014). This requires the following norm equivalences: for each  $\phi : \Omega \rightarrow \mathbb{R}$

$$\|\hat{\phi}\|_{H^1} \cong \|\hat{\phi} \circ G^{-1}\|_{H^1} = \|\phi\|_{H^1} \quad (2.14)$$

$$\|\hat{\phi}\|_{L^2} \cong \|\hat{\phi} \circ G^{-1}\|_{L^2} = \|\phi\|_{L^2}. \quad (2.15)$$

The equivalences above depend on the regularity of  $G$ . In particular they are implied by the following assumptions:

- $G : \hat{\Omega} \rightarrow \Omega$  is a homeomorphism: a continuous bijective map with continuous inverse.
- Both  $\text{grad } G$  and  $\text{grad } G^{-1}$  are continuous and bounded on their respective domains,
- The discrete spaces have the same (or lower) local smoothness than the parametrization  $G$ . This means that the spline spaces must allow for discontinuities of the  $k$  derivatives at  $\mathbf{x}$  if  $G$  is not  $C^k$  at  $\mathbf{x}$ .

The second condition can be weakened to piecewise continuity of  $\text{grad} G$  and  $\text{grad} G^{-1}$ , but then the inf-sup-stability must be proved on each domain on which these are continuous. This allows to consider multipatch domains.

### 2.3 Discrete inf-sup

The inf-sup stability of the Taylor-Hood discretizations in FEM was studied by Engelman & Bercovier (1979); Verfürth (1984) and Stenberg (1990). Stenberg combined the previous results and proved the stability of the generalized Taylor-Hood elements. The proof is based on the inf-sup stability of the pair  $V, P$ , an approximation operator  $\wp: V \rightarrow V_h$ , the trick from Verfürth (1984) and the *macro-element* technique.

LEMMA 2.1 (Verfürth's trick) The inf-sup condition (ii) for the pair  $V_h, P_h$  is implied by the following two auxiliary conditions:

$$\forall q \in P_h, \exists \mathbf{w}_1 \in V_h : b(\mathbf{w}_1, q) \geq (c_1 - \eta(q)) \|\mathbf{w}_1\|_V \|q\|_P \quad (2.16)$$

$$\forall q \in P_h, \exists \mathbf{w}_2 \in V_h : b(\mathbf{w}_2, q) \geq c_2 \eta(q) \|\mathbf{w}_2\|_V \|q\|_P \quad (2.17)$$

where  $c_1, c_2 > 0$  and  $\eta(q) \geq 0$ .

*Proof.* By choosing either  $\mathbf{w} = \mathbf{w}_1$  or  $\mathbf{w} = \mathbf{w}_2$  it follows that (ii) holds for the pair  $V_h, P_h$  with

$$\beta_h \geq \min_{q \in P_h} \max\{c_1 - \eta(q), c_2 \eta(q)\} \geq \min_{t \in \mathbb{R}_+} \max\{c_1 - t, c_2 t\} = \frac{c_1 c_2}{1 + c_2}. \quad (2.18)$$

□

LEMMA 2.2 (approximation) The condition in (2.16) holds with

$$c_1 = \frac{\beta}{\|\wp\|} \quad \text{and} \quad \eta(q) = \frac{\|\rho \text{grad} q\|_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)}} \quad (2.19)$$

provided that there exists  $\wp: V \rightarrow V_h$  such that

$$\|\rho^{-1}(\mathbf{w} - \wp \mathbf{w})\|_{L^2(\Omega)} \leq |\mathbf{w}|_{H^1(\Omega)} \quad (2.20)$$

where  $\rho: \Omega \rightarrow \mathbb{R}$  is a multiple of the mesh size: on each element  $E$ ,  $\rho(E) = cR(E)$  for some  $c > 0$ .

LEMMA 2.3 (macro-elements) The condition in (2.17) holds for  $\eta(q)$  as in (2.19) with constant

$$c_2 \geq \frac{c_{\text{SD}}}{m} \quad (2.21)$$

if the domain can be covered by subdomains  $\{M_1, \dots, M_N\}$  such that

- at most  $m$  subdomains overlap (each point  $p \in \Omega$  is contained in at most  $m$  sets  $\text{int} M_i$ );
- on each  $M_i$  the following local inf-sup condition holds:

$$\forall q \in P_h, \exists \mathbf{w}_i \in V_h \cap H_0^1(M_i)^n : b(\mathbf{w}_i, q) \geq c_{\text{SD}} \|\rho \text{grad} q\|_{L^2(M_i)} |\mathbf{w}_i|_V. \quad (2.22)$$

THEOREM 2.3 If there exists a continuous approximation operator  $\wp: V \rightarrow V_h$  as in Lemma 2.1 and a subdomain covering of  $\Omega$  as in Lemma 2.3, then the inf-sup condition holds with

$$\beta_h \geq \frac{\beta c_{\text{SD}}}{(m + c_{\text{SD}}) \|\wp\|} > 0. \quad (2.23)$$

*Proof.* Combine Lemmas 2.1, 2.2 and 2.3. □

The strategy to use Theorem 2.3 effectively is to consider subdomains that are images through a similarity of *macro-elements* in some catalog  $\mathfrak{C}$  of stable macro-elements. Each macro-element in  $\mathfrak{C}$  is a triplet  $(M_\sigma, V_\sigma, P_\sigma)$ , where  $M_\sigma \subset \mathbb{R}^n$ ,  $V_\sigma \subset H_0^1(M_\sigma)^n$  is a velocity space and  $P_\sigma \subset L^2(M_\sigma)$  is a pressure space. A macro-element is stable if exists  $C_\sigma > 0$  such that:

$$\forall q \in P_\sigma, \exists \mathbf{w} \in V_\sigma : b(\mathbf{w}, q) \geq C_\sigma \|\rho \operatorname{grad} q\|_{L^2(\mathbb{R}^n)} |\mathbf{w}|_{H^1(\mathbb{R}^n)}.$$

The macro-elements in the catalog  $\mathfrak{C}$  are indexed by a shape parameter  $\sigma \in \mathbb{R}^s$ . The constant  $C_\sigma$  is preserved under similarities and depends continuously on  $\sigma$ . If the shape parameter  $\sigma$  varies in a compact set  $\mathfrak{S}$ , then Theorem 2.3 applies and condition (ii) holds with

$$\beta_h \geq \frac{\beta C_{\mathfrak{S}}}{(m + C_{\mathfrak{S}}) \|\varrho\|} \quad (2.24)$$

where  $C_{\mathfrak{S}} = \min_{\sigma \in \mathfrak{S}} \{C_\sigma\}$ .

The macro-element collection approach does not apply directly to IGA because of the distortion caused by the parametrization of the domain. The solution is to use a macro-element collection  $\mathfrak{C}$  on the parametric domain  $\hat{\Omega}$  and to transfer the result to the physical macro-elements. The effect of the mapping on the inf-sup was studied by Bressan (2011) by decomposing  $G$  in an affine part  $F$  and a reminder term  $T$ :

$$G(\hat{\mathbf{x}}) = F(\hat{\mathbf{x}}) + T(\hat{\mathbf{x}}). \quad (2.25)$$

$F(\hat{\mathbf{x}})$  is given by the averages of  $G$  and  $\operatorname{grad} G$  on the parametric macro-element  $\hat{M}$ . It preserves the inf-sup up to a factor that depends on the regularity of  $G$ . The reminder  $T$  can be bounded in terms of the size of the parametric macro-element  $\hat{M}$ .

The compactness of  $\mathfrak{S}$  is obtained by requiring *shape-regularity* of the elements of the parametric macro-elements (and thus of the mesh). The shape regularity of a set  $S$  is the ratio between its radius  $R(S)$  and that of the biggest in-circle  $r(S)$ . A mesh  $\mathcal{M}$  is *shape regular* if there is  $C_{\text{shape}}$  such that for all elements  $E \in \mathcal{M}$

$$\frac{R(E)}{r(E)} \leq C_{\text{shape}}. \quad (2.26)$$

A consequence of shape-regularity is *local quasi-uniformity* of the macro-elements. A mesh  $\mathcal{M}$  is *locally quasi-uniform* if there is  $C_{\text{uniform}}$  such that for each pair of neighbouring elements  $E_1, E_2 \in \mathcal{M}$

$$\frac{R(E_1)}{R(E_2)} \leq C_{\text{uniform}}. \quad (2.27)$$

As seen later, local quasi-uniformity is required for the construction of the approximation operator presented here.

LEMMA 2.4 (Bressan (2011)) The local inf-sup condition of Lemma 2.3 for the pair  $V_h, P_h$  holds with a constant  $C_{\mathfrak{S}, G}$  provided that

- the parametric mesh can be covered with parametric macro-elements of a stable collection  $\mathfrak{C}$  whose shape parameter is in the compact set  $\mathfrak{S}$ ;
- the parametric macro-element diameters are smaller than a constant  $h_{\mathfrak{S}, G}$  depending on  $\mathfrak{S}$  and  $G$ .

Bressan & Sangalli (2013) proposed two types of stable parametric macro-elements: the Taylor-Hood type and the SubGrid type. Both depend on four parameters: the degrees and smoothnesses of the velocity and pressure. These parameters will be denoted according to the following table:

	$V_\sigma$	$P_\sigma$
degree	$d_V$	$d_P$
smoothness	$s_V$	$s_P$

In both types the equality  $d_V = d_P + 1$  ensures the same order of the approximation error for the velocity and for the pressure space, so this is the preferred choice. In both cases the shape parameter  $\sigma$  contains the normalized knot-values of the knot vectors associated to the tensor-product spline space used to discretize the components of the velocity space.

TH-type macro-elements are stable provided that

$$d_V - s_V > d_P - s_P.$$

A parametric macro-element is described by  $n(L+1)$ -tuples of break points

$$\Theta_i = [\theta_{i,1}, \dots, \theta_{i,L+1}], \quad i = 1, \dots, n$$

with  $\theta_{1,1} = 0, \theta_{1,L+1} = 1$  where

$$L = \left\lceil \frac{s_V + s_P + 2}{(d_V - s_V) - (d_P - s_P)} \right\rceil. \quad (2.28)$$

The knot vectors of the tensor-product spline spaces for the velocity and pressure are obtained by repeating the break points according to  $d_V, s_V, d_P, s_P$ . The multiplicity of each break point in the velocity knot vectors is  $d_V - s_V$ . The multiplicity of internal break point in the pressure knot vectors is  $d_P - s_P$  and that of end knots is  $d_P + 1$ . The shape parameter is

$$\sigma = (\theta_{1,2}, \dots, \theta_{1,L}, \theta_{2,1}, \dots, \theta_{n,L+1}).$$

SG-type macro-elements are stable provided that

$$2(d_V - s_V) > d_P - s_P.$$

Again a parametric macro-element is described by  $n(L+1)$ -tuples of break points with

$$L = 2 \left\lceil \frac{s_V + s_P + 2}{2(d_V - s_V) - (d_P - s_P)} \right\rceil. \quad (2.29)$$

The knot vectors of the velocity space are obtained as in the TH-case. Those of the pressure space by using every other break point so that every element of the pressure contains  $2^n$  elements of the velocity. The shape parameter is as in the TH-type. The SG-type macro-elements can achieve maximal smoothness for both the velocity and the pressure spaces.

### 3. Application to hierarchical splines

Hierarchical spline spaces were introduced by Forsey & Bartels (1988). They have been studied extensively by Giannelli *et al.* (2012); Giannelli & Jüttler (2013b); Mokriš *et al.* (2014); Giannelli *et al.* (2014); Berdinsky *et al.* (2014); Giannelli & Jüttler (2013a). The application of hierarchical splines

to IGA is under study by many authors Vuong *et al.* (2011); Kuru *et al.* (2014); Bornemann & Cirak (2013); Buffa & Giannelli (2016); Buffa *et al.* (2016).

The main idea of hierarchical spline basis is to enrich a tensor-product spline basis by adding functions from finer spaces while, simultaneously, removing redundant functions. Let  $\widehat{S}_0, \dots, \widehat{S}_\ell$  be a sequence of tensor-product spline spaces (of the same degree) such that  $\widehat{S}_i \subseteq \widehat{S}_{i+1}$  and  $\mathfrak{B}_0, \dots, \mathfrak{B}_\ell$  be the corresponding bases. Let

$$\widehat{\Omega}_0 = \widehat{\Omega} \supseteq \dots \supseteq \widehat{\Omega}_\ell \supseteq \widehat{\Omega}_{\ell+1} = \emptyset$$

be a sequence of domains such that for  $i = 1, \dots, \ell$  the set  $\widehat{\Omega}_i$  is a union of elements of the mesh  $\widehat{\mathcal{M}}_{i-1}$  associated to  $\widehat{S}_{i-1}$ . The hierarchical basis  $\widehat{\mathcal{H}}$  is defined by

$$\widehat{\mathcal{H}} = \{\widehat{\psi} : \widehat{\psi} \in \widehat{\mathfrak{B}}_i, \text{supp } \widehat{\psi} \subseteq \widehat{\Omega}_i \text{ and } \text{supp } \widehat{\psi} \not\subseteq \widehat{\Omega}_{i+1}, i = 0, \dots, \ell\}. \quad (3.1)$$

and the hierarchical space is  $\widehat{\mathbb{H}} = \text{span } \widehat{\mathcal{H}}$ . The *truncated* basis  $\widehat{\mathcal{H}}_T$  is a modification of the standard basis. Each function  $\widehat{\psi} \in \widehat{\mathcal{H}}$  is replaced by its truncated version  $\widehat{\psi}_T$ . The truncated basis has two important properties: it is a partition of unity and it preserves polynomial coefficients. This means that the coefficients of the expansion of a polynomial  $\widehat{g}$  in the truncated basis are the same as with respect of the tensor bases:

$$\forall i = 0, \dots, \ell, \quad \widehat{g} = \sum_{\widehat{\psi} \in \widehat{\mathfrak{B}}_i} c_{\widehat{\psi}} \widehat{\psi} \quad \Rightarrow \quad g = \sum_{\widehat{\psi}_T \in \widehat{\mathcal{H}}_T} c_{\widehat{\psi}} \widehat{\psi}_T. \quad (3.2)$$

This property is important in the construction of the approximation operators. In the following  $\widehat{\mathcal{M}}$  and  $\mathcal{M}$  will refer to the parametric and physical meshes, i.e., to the collection of the elements of the space.

### 3.1 Approximation operator for hierarchical splines

We describe an approximation operator  $\wp$  that fulfills the requirements of Lemma 2.2. The operator is defined on the parametric domain and then it is pushed to the physical domain:

$$\wp(\phi) = \widehat{\wp}(\widehat{\phi}) \circ G^{-1} = \widehat{\wp}(\phi \circ G) \circ G^{-1}.$$

The approximation operators on spline spaces are often described by providing the coefficients with respect to a basis. To avoid heavy notation we set the analysis directly in the physical domain and we detail the construction for scalar valued functions; the extension to vector valued functions is obtained by applying the operator component-wise.

Let  $\mathcal{H}_T$  be the push-forward of the truncated basis, then  $\wp : H^1(\Omega) \rightarrow \mathbb{H}$  can be written as

$$\wp\phi = \sum_{\psi \in \mathcal{H}_T} \lambda_\psi(\phi) \psi, \quad (3.3)$$

where the  $\lambda_\psi$  are in  $H^1(\Omega)^*$ . Speleers & Manni (2016) showed that a set of  $\widehat{\lambda}_\psi$  for hierarchical bases can be constructed from those for the spaces  $\widehat{S}_0, \dots, \widehat{S}_\ell$ . Moreover following their construction the following properties hold (rewritten in the physical domain):

- $\wp$  reproduces constant functions:  $\forall g$  constant function,  $\sum_{\psi \in \mathcal{H}_T} \lambda_\psi(g) \psi = g$ .



- The operators  $\lambda_\psi$  can be represented as function in  $L^2(\Omega)$  with local support and bounded norms. More precisely with support included in  $\text{supp } \psi$  and norm

$$\|\lambda_\psi\|_{L^2(\Omega)} \leq C|\text{supp } \psi|^{-1/2}.$$

The above imply the quasi-optimal approximation estimates with respect to  $L^\infty$  norm (Speleers & Manni, 2016, Theorem 6). Here the result is rewritten for the Sobolev norms of interest, and a global estimate is derived as a corollary.

LEMMA 3.1 If  $\mathcal{M}$  is shape-regular the approximation error of  $\mathcal{I}$  on a mesh element  $K$  is bounded by

$$\|\phi - \mathcal{I}\phi\|_{L^2(K)} \leq C_{\text{app}} h_K |\phi|_{H^1(\tilde{K})}, \quad (3.4)$$

where:

$\tilde{K}$  is the union of the supports of the basis functions in  $\mathcal{H}_T$  that are active on  $K$ ;

$C_{\text{app}}$  is a constant that depends on the regularity of the parametric mesh (2.26), (2.27) and of the parametrization  $G$ .

*Proof.* From the reproduction of constants we deduce

$$\begin{aligned} \|\mathcal{I}\phi - \phi\|_{L^2(K)} &= \|\mathcal{I}(\phi - c) - (\phi - c)\|_{L^2(K)} \\ &\leq \inf_{c \in \mathbb{R}} \left( \|\mathcal{I}(\phi - c)\|_{L^2(K)} + \|\phi - c\|_{L^2(K)} \right) \end{aligned} \quad (3.5)$$

then, since  $\mathcal{I}(\phi - c)|_K$  depends only on  $(\phi - c)|_{\tilde{K}}$ ,

$$\begin{aligned} \|\mathcal{I}\phi - \phi\|_{L^2(K)} &\leq (\|\mathcal{I}\| + 1) \inf_c \|\phi - c\|_{L^2(\tilde{K})} \\ &\leq (\|\mathcal{I}\| + 1) C_P(\tilde{K}) h_{\tilde{K}} |\phi|_{H^1(\tilde{K})}, \end{aligned} \quad (3.6)$$

where:

$C_P(\tilde{K})$  is the Poincaré constant of  $\tilde{K}$  and depends on the shape of  $\tilde{K}$ ;

$h_{\tilde{K}}$  is the diameter of  $\tilde{K}$ ;

$\|\mathcal{I}\|$  is the norm of  $\mathcal{I}$  as an operator  $L^2(\tilde{K}) \rightarrow L^2(K)$ .

Eq. (3.4) follows from Eq. (3.6) if  $C_P(\tilde{K}) h_{\tilde{K}} \cong h_K$  and  $\|\mathcal{I}\|$  is uniformly bounded. The former is implied by the regularity assumptions on  $G$  and the mesh. The latter is a consequence of the assumptions on the  $\lambda_\psi$  and of the partition of unity property of the truncated basis.  $\square$

COROLLARY 3.1 The approximation error of  $\mathcal{I}$  is bounded by:

$$\|\rho(\phi - \mathcal{I}\phi)\|_{L^2(\Omega)}^2 \leq |\phi|_{H^1(\Omega)}, \quad (3.7)$$

where  $\rho = C_{\text{app}}^{-1} h_K^{-1} C_{\text{over}}^{-1/2}$  and  $C_{\text{over}} = \max_{K \in \mathcal{M}} \#\{\psi \in \mathcal{H} : \text{supp } \psi \supseteq K\} \max_{\psi \in \mathcal{H}_T} \#\{K \subseteq \text{supp } \psi\}$ .

*Proof.* Rewrite (3.4) as  $\|C_{\text{app}}^{-1}h_K^{-1}(\phi - \wp\phi)\|_{L^2(K)}^2 \leq |\phi|_{H^1(\tilde{K})}^2$  and sum over all elements in the mesh

$$\begin{aligned}
\|C_{\text{app}}^{-1}h_K^{-1}(\phi - \wp\phi)\|_{L^2(\Omega)}^2 &= \sum_{K \in \mathcal{M}} \|C_{\text{app}}^{-1}h_K^{-1}(\phi - \wp\phi)\|_{L^2(K)}^2 \\
&\leq \sum_{K \in \mathcal{M}} |\phi|_{H^1(\tilde{K})}^2 = \sum_{K \in \mathcal{M}} \sum_{\substack{Q \in \mathcal{M}: \\ Q \subseteq \tilde{K}}} |\phi|_{H^1(Q)}^2 \\
&= \sum_{Q \in \mathcal{M}} \#\{K \in \mathcal{M} : Q \subseteq \tilde{K}\} |\phi|_{H^1(Q)}^2 \\
&\leq \sum_{Q \in \mathcal{M}} \#\{i : Q \subseteq \text{supp } \psi_i\} \#\{K \subseteq \text{supp } \psi_i\} |\phi|_{H^1(Q)}^2 \\
&\leq C_{\text{cover}} |\phi|_{H^1(\Omega)}^2
\end{aligned} \tag{3.8}$$

□

In the common case of levels corresponding to dyadic refinement in the parametric domain the mesh regularity assumptions are equivalent to the fact that the boundaries of  $\Omega_i$  and  $\Omega_{i+j}$  are disjoint for  $j > \log_2(C_{\text{uniform}})$ . Unfortunately this is not sufficient for a common upper bound of  $C_{\text{cover}}$ . All the required conditions are implied by *admissibility* of the mesh as defined by Buffa & Giannelli (2016).

### 3.2 TH- and SG-type methods with hierarchical splines

Let  $\widehat{V}_0 \subset \dots \subset \widehat{V}_\ell$  and  $\widehat{P}_0 \subset \dots \subset \widehat{P}_\ell$  be two hierarchies of tensor-product spline spaces on  $\widehat{\Omega}$  such that the pairs  $V_i, P_i$  are either all of type TH or all of type SG. Then let  $\widehat{\mathbb{H}}_V$  and  $\widehat{\mathbb{H}}_P$  be respectively the hierarchical spline spaces associated spaces  $\{\widehat{V}_i\}$  and  $\{\widehat{P}_i\}$  and a common hierarchy of nested domains  $\widehat{\Omega}_0 \supseteq \dots \supseteq \widehat{\Omega}_\ell \subseteq \widehat{\Omega}_{\ell+1} = \emptyset$ . The discretization spaces are the push-forward of  $\widehat{\mathbb{H}}_V$  and  $\widehat{\mathbb{H}}_P$ :

$$\begin{aligned}
V_h &= \mathbb{H}_V^n \cap V = \{\widehat{\mathbf{w}} \circ G^{-1} : \widehat{\mathbf{w}} \in \widehat{\mathbb{H}}_V^n\} \cap V, \\
P_h &= \mathbb{H}_P \cap P = \{\widehat{q} \circ G^{-1} : \widehat{q} \in \widehat{\mathbb{H}}_P\} \cap P.
\end{aligned} \tag{3.9}$$

For these spaces stability follows from Theorem 2.3 if  $\Omega$  can be covered by macro-elements.

**THEOREM 3.1** If the parametric mesh can be covered by macro-elements of type SG or TH in a collection  $\mathfrak{S}$  defined by shape regularity and local quasi-uniformity and such that the diameter of the covering macro-elements is less than  $h_{\mathfrak{S}, G}$  (see Lemma 2.4) then the method is inf-sup stable and the inf-sup constant  $\beta_h$  depends only on  $\mathfrak{S}$ ,  $G$  and  $C_{\text{cover}}$ . Consequently the a-priori estimates for the error (2.10) and (2.11) hold.

The request of a macro-element covering constrains the class of meshes to which the result can be applied. Indeed, TH and SG macro-elements have a tensor-product structure and thus it is necessary that each ring

$$\Delta_i = \Omega_i \setminus \Omega_{i+1} \tag{3.10}$$

is covered separately. In practice, given a sequence of domains  $\Omega'_0 \supseteq \dots \supseteq \Omega'_\ell \supseteq \Omega'_{\ell+1} = \emptyset$  that does not satisfy the condition above, for instance the result of a refinement step, it is possible to construct a corresponding sequence  $\Omega_0 \supseteq \dots \supseteq \Omega_\ell \supseteq \Omega_{\ell+1} = \emptyset$  that satisfies Theorem 3.1.

A possible solution for dyadic refinement is to associate to each level  $i$  a partition  $\mathcal{P}_i$  of  $\Omega$  into non-overlapping closed macro-elements. The different partitions must be compatible, meaning that each macro-element in  $\mathcal{P}_i$  is the union of  $2^n$  macro-elements in  $\mathcal{P}_{i+1}$ .

LEMMA 3.2 For any  $\Omega'_0 \supseteq \dots \supseteq \Omega'_\ell$  the sequence defined as follows. For  $i = 1, \dots, \ell$

$$\Omega_i = \mathcal{E}_{i-1}(\Omega_{i+1} \cup \Omega'_i) \quad (3.11)$$

where  $\mathcal{E}_i(Q) = \bigcup\{M \in \mathcal{P}_i : M \cap \bar{Q} \neq \emptyset\}$  has rings  $\Delta_i = \Omega_i \setminus \Omega_{i+1}$  that can be covered by macro-elements from  $\mathcal{P}_i$ . Moreover each element of  $\mathcal{M}$  is contained in exactly one macro-element ( $m = 1$  in Lemma 2.3).

*Proof.* By construction both  $\Omega_i$  and  $\Omega_{i+1}$  are unions of macro-elements of  $\mathcal{P}_i$ . Since the macro-elements are non-overlapping then  $\Delta_i$  is covered by the

$$\bigcup\{M \in \mathcal{P}_i : M \subseteq \Omega_i \text{ and } M \not\subseteq \Omega_{i+1}\}.$$

□

The construction in Lemma 3.2 ensures that on each element  $K$  of level  $i$  are active only functions  $\psi \in \mathfrak{B}_j$  with  $j = i, i-1$ . This means that the resulting mesh is *2-admissible* according to Buffa *et al.* (2016). Moreover, in the common case of dyadic refinement the construction implies both shape-regularity and local-quasi-uniformity.

In conclusion, we showed that it is possible to construct and refine a pair of discretization spaces in such a way that the hypotheses of Theorem 3.1 are fulfilled with a common lower bound on  $\beta_h$ . Thus the convergence discretization error is controlled by the best approximation error up to a constant factor.

#### 4. Numerical stability tests

The inf-sup constant  $\beta_h$  can be computed numerically by solving the following eigenproblem:

$$B_h N_V^{-1} B_h^T q = \lambda N_P q, \quad (4.1)$$

where  $B_h : V_h \rightarrow P_h^*$ ,  $N_V : V_h \rightarrow V_h^*$  and  $N_P : P_h \rightarrow P_h^*$  are respectively the matrices corresponding to  $b_h$ , the scalar product on  $V$  and the scalar product on  $P$ . Let  $\lambda_{\min}$  be the smallest eigenvalue of problem (4.1) then, as described in (Brezzi & Fortin, 1991),

$$\beta_h = \sqrt{\lambda_{\min}}.$$

The tests focus on the the dependence of  $\beta_h$  on the polynomial degree ( $d_P$ ), on the number of levels of the hierarchical space ( $\ell$ ), on the type of discretization (TH or SG) and on the hypothesis of Theorem 3.1. In particular  $\beta_h$  is computed on meshes both fulfilling and not fulfilling the hypothesis of Theorem 3.1. In all tests the domain is  $\Omega = \widehat{\Omega} = [0, 1]^2$  parametrized by the identity, the spaces are balanced and of maximum smoothness ( $d_V = d_P + 1$ ,  $s_P = d_P - 1$ ) and the refinement of the spaces  $S_i$  is dyadic.

##### 4.1 Macro-element inf-sup

As a first test we computed the inf-sup on a single 2D macro-element. The results can be read in Table 1. We observe that the inf-sup constant  $\beta_h$  decreases fast as a function of the degree (and the smoothness). The result can be interpreted by observing that on the macro-element the velocities are constrained to decrease to zero with  $C^{s_V}$  smoothness toward the boundary. This condition is unnatural in  $H^1(\Omega)^n$  and can be enforced only on the discrete space. The effect of the degree and of the smoothness can be made clearer by the following 1D example.

$d_p$	2	3	4	5	6
TH	$4.68 \cdot 10^{-1}$	$4.86 \cdot 10^{-2}$	$4.96 \cdot 10^{-4}$	$1.72 \cdot 10^{-6}$	$3.23 \cdot 10^{-9}$
SG	$1.73 \cdot 10^{-1}$	$2.98 \cdot 10^{-3}$	$8.90 \cdot 10^{-6}$	$1.06 \cdot 10^{-8}$	$7.00 \cdot 10^{-12}$

Table 1. Numerically computed inf-sup on 2D macro-elements for various pressure degrees  $d_p$ ,  $d_V = d_p + 1$ . For macro-elements of type TH both pressure and velocity are  $C^{d_p-1}$  while for SG-type the pressure is  $C^{d_p-1}$  and velocity  $C^{d_p}$ .

$d_p$	2	3	4	5	6	7	8
2	0.4600	0.4574	0.4568	0.4562	0.4562	0.4562	0.4562
3	0.4537	0.4521	0.4514	0.4511	0.4508	0.4508	0.4508
4	0.4500	0.4485	0.4479	0.4477	0.4475	0.4475	0.4475

Table 2. Computed inf-sup constant on graded mesh for the TH method depending on degree and number of levels. The parameters are  $d_V = d_p + 1, s_V = s_p = d_p - 1$ .

EXAMPLE 4.1 Consider a 1D element of length 1,  $K = [0, 1]$ . Now let  $M = [0, L]$  be a macro-element and let

$$q = \begin{cases} (1-x)^{d_p} & x \leq 1 \\ 0 & x > 1 \end{cases}$$

be the pressure. Then

$$b(w, q) = cd_p \binom{d_p + r_V + 1}{d_p}^{-1},$$

where  $c$  is the coefficient of  $x^{r_V}$  in  $w|_{[0,1]}$ . This is not compensated by the behavior of the norms:

$$\|\rho \text{grad } q\|_{L^2(M)} = d_p(2d_p - 1)^{-1/2}, \quad \|w\|_{L^2(M)} \geq \|w\|_{L^2([0,1])} = c(2r_V + 1)^{-1/2}.$$

#### 4.2 Graded meshes

The meshes of this test are refined at the center in such a way that the  $\Omega_i$  are almost-concentric squares and the  $\Delta_i = \Omega_i \setminus \Omega_{i+1}$  are rings with a thickness of at least  $2d_p + 1$  pressure elements. In this way each  $\Delta_i$  can be covered with macro-elements. See the meshes in Fig. 1. The results are reported in Table 2 for the TH element and in Table 3 for the SG element. Observe that the discrete inf-sup constant is independent of the number of level as predicted by the theory. An interesting fact is that the computed discrete inf-sup constants exceed the continuous inf-sup constant that, according to Costabel & Dauge (2015); Costabel *et al.* (2015), is contained in

$$\left[ \sin\left(\frac{\pi}{8}\right), \sqrt{\frac{1}{2} - \frac{1}{\pi}} \right].$$

#### 4.3 Non macro-element-covered meshes

The meshes of this test are refined similarly to those in the previous test, but the thickness of the  $\Delta_i$  is not enough to allow for a macro-element covering. See Figure 2. As a consequence Theorem 3.1 does

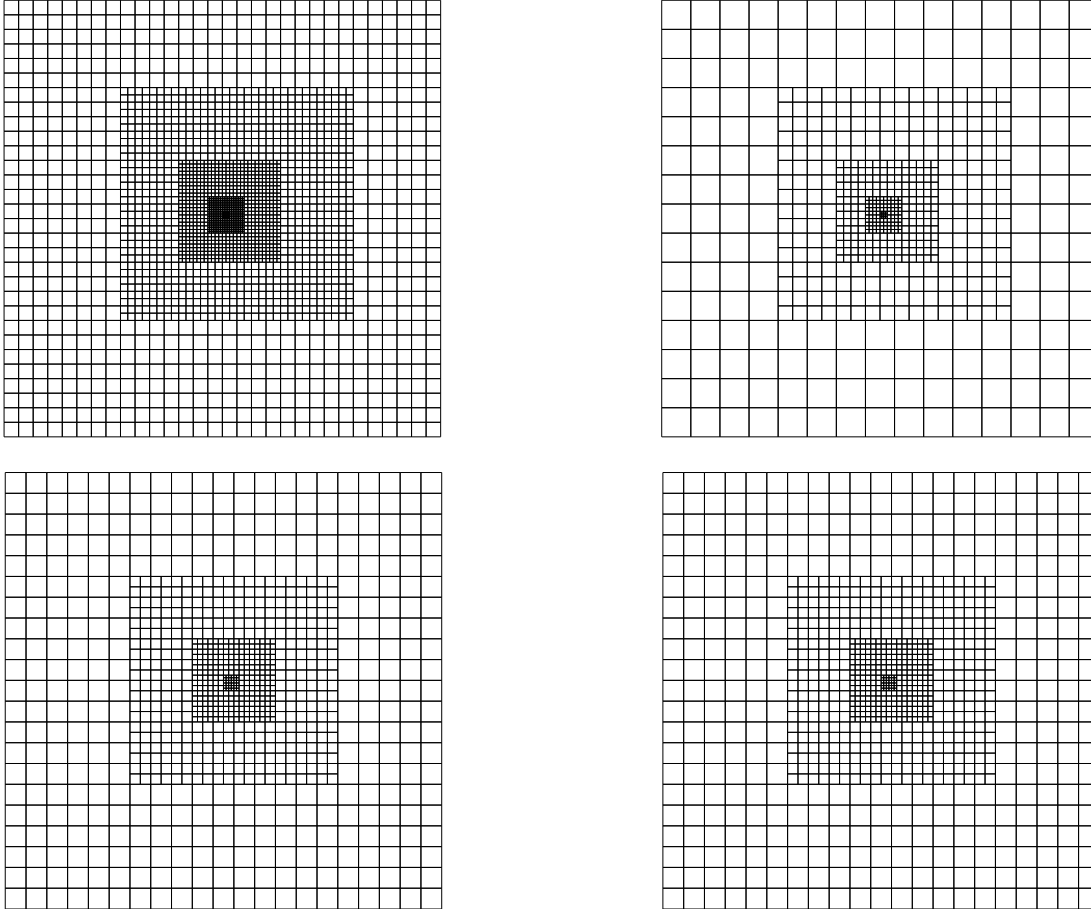


Figure 1. Graded meshes. First row: the meshes of velocity (left) and pressure (right) for the SG method with parameters  $d_v = 3, d_p = 2, s_v = 2, s_p = 1$  and 5 levels. Second row: the meshes of velocity (left) and pressure (right) for the TH method with parameters  $d_v = 4, d_p = 3, s_v = s_p = 2$  and 4 levels.

$d_p$	2	3	4	5	6	7	8
2	0.4664	0.4632	0.4623	0.4616	0.4616	0.4616	0.4616
3	0.4578	0.4558	0.4549	0.4546	0.4542	0.4542	0.4542
4	0.4531	0.4513	0.4506	0.4504	0.4502	0.4502	0.4502

Table 3. Computed inf-sup constant on graded mesh for the SG method depending on degree and number of levels. The parameters are  $d_v = d_p + 1, s_v = d_p, s_p = d_p - 1$ .

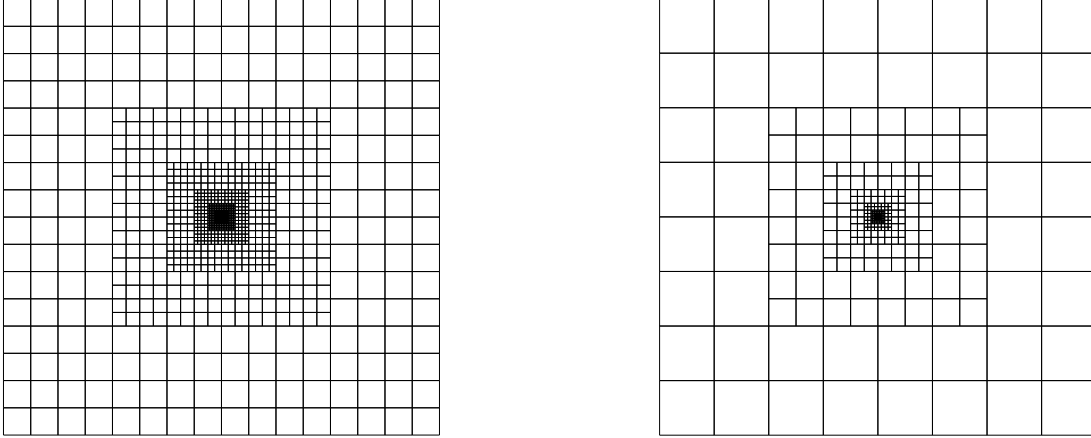


Figure 2. Non-macro-elements-covered meshes. On the left the velocity mesh for the SG method with parameters  $d_v = 5, d_p = 4, s_v = 4, s_p = 3$  and 7 levels; on the right the velocity mesh for TH method with the same parameters.

$d_p$	2	3	4	5	6	7	8
2	0.4705	0.4705	0.4705	0.4705	0.4705	0.4705	0.4705
3	0.4604	0.4593	0.4593	0.4593	0.4593	0.4593	0.4593
4	0.4564	0.4564	0.4564	0.4564	0.4564	0.4564	0.4564

Table 4. Computed inf-sup constant on non-macro-elements-covered meshes for the TH method depending on degree and number of levels. The parameters are  $d_v = d_p + 1, s_v = s_p = d_p - 1$ .

not apply. The results are reported in Table 4 for the TH element and in Table 5 for the SG element. Surprisingly the inf-sup constant in this case is greater (and thus better) than in the previous case. Our interpretation is that this refinement increases the dimension of the velocity space compared to that of the pressure space even more, hence the inf-sup constant increases.

#### 4.4 Non-graded meshes

In this test the meshes are refined only in half of the domain so that ratio between the size of neighbouring elements increases geometrically on the number of levels (Fig. 3). In this case, not only the

$d_p$	2	3	4	5	6	7	8
2	0.4817	0.4817	0.4817	0.4817	0.4817	0.4817	0.4817
3	0.4661	0.4645	0.4645	0.4645	0.4645	0.4645	0.4645
4	0.4610	0.4610	0.4610	0.4610	0.4610	0.4610	0.4610

Table 5. Computed inf-sup constant on non-macro-elements-covered graded meshes for the SG method depending on degree and number of levels. The parameters are  $d_v = d_p + 1, s_v = d_p, s_p = d_p - 1$ .

$d_p$	2	3	4	5	6	7	8
1	0.4683	0.4716	0.4690	0.4631	0.4627	0.4623	0.4621
2	0.4685	0.4728	0.4713	0.4629	0.4628	0.4559	0.4559
3	0.4757	0.4669	0.4599	0.4599	0.4536	0.4536	0.4536
4	0.4557	0.4598	0.4570	0.4567	0.4516	0.4516	0.4516

Table 6. Computed inf-sup constant on non-graded meshes for the TH method depending on degree and number of levels. The parameters are  $d_V = d_p + 1, s_V = s_p = d_p - 1$ .

$d_p$	2	3	4	5	6	7	8
1	0.5256	0.4981	0.4985	0.4804	0.4804	0.4804	0.4804
2	0.4952	0.4827	0.4827	0.4703	0.4703	0.4614	0.4614
3	0.4901	0.4759	0.4648	0.4651	0.4577	0.4577	0.4577
4	0.4764	0.4707	0.4614	0.4615	0.4551	0.4551	0.4551

Table 7. Computed inf-sup constant on non-graded meshes for the SG method depending on degree and number of levels. The parameters are  $d_V = d_p + 1, s_V = d_p, p = d_p - 1$ .

macro-element covering is missing, but also the approximation properties of  $\mathcal{S}$  deteriorates as the number of levels increases. The results are presented in the tables 6 and 7. Again we observe that the inf-sup constant is stable on both degree and the number of levels.

#### 4.5 Observation on the numerically computed inf-sup constants

The computed inf-sup constant on macro-elements shows that the lower bound obtained by the theory is very pessimistic. The strong dependence on the smoothness is not observed and the lack of a macro-element cover does not influence the inf-sup constant. Instead, the computed inf-sup constant of the TH- and SG-type methods are close to the continuous inf-sup constant, sometimes even greater, regardless of the degree, the mesh grading and the macro-element covering.

## 5. Adaptive implementation

The main motivation for the extension of the inf-sup stability to hierarchical splines is adaptivity. In this section we report a numerical experiment based on the TH adaptive IGA scheme. We solved a 2D flow inside a T-shaped pipe (Fig. 4). The forcing term is constant and directed along the  $x$  axis. The homogeneous Dirichlet condition is imposed on the boundary except for the inlet  $\Gamma_{\text{in}}$  and the outlet  $\Gamma_{\text{out}}$  where homogeneous Neumann conditions are enforced.

The regularity of the continuous solution is ruined by the presence of two concave corners in the domain. More precisely, the solution can be decomposed in a regular part  $\mathbf{u}_{\text{reg}}$ , whose regularity depends only on  $(\mathbf{f}$  and  $g$ ), and a singular part  $\mathbf{u}_{\text{sing}}$ , whose regularity depends on the geometry of the domain. The behaviour of the singular part in the neighbourhood a concave corner is described by

$$\mathbf{u}_{\text{sing}} = \sum_{q=0}^Q r^\lambda \psi_q(\theta) \log(r)^q, \quad (5.1)$$

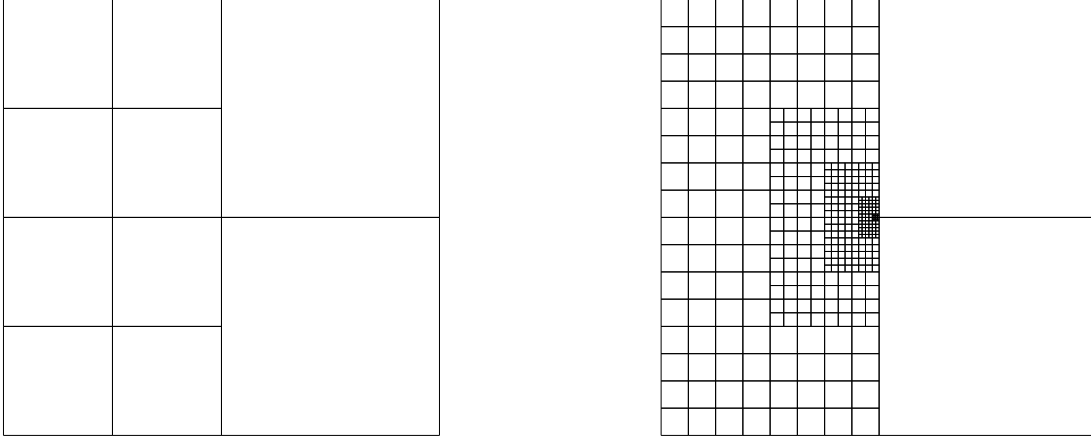


Figure 3. Non graded meshes. Pressure meshes for  $d_p = 2, s_p = 1$  with 2 levels (left), and 8 levels (right).

where  $\lambda$  and  $Q$  can be computed from the geometry (Dauge, 1989). In the test case (a corner singularity in 2D with an angle of  $\frac{3}{2}\pi$ ) the exponent  $\lambda$  is the smallest positive solution of

$$\sin\left(\frac{3}{2}\pi\lambda^2\right) - \lambda^2 = 0 \quad (5.2)$$

which approximately equals 0.54448.... This means that the solution  $(\mathbf{u}, p)$  is at most in  $H^{\lambda+1}(\Omega)^2 \times H^\lambda(\Omega)$  because higher order derivatives are not square-integrable. Consequently the highest order of convergence that can be achieved with uniform refinement is

$$\text{dofs}^{-\frac{\lambda}{2}} \approx \text{dofs}^{-0.272\dots}$$

The domain is represented by a 4-patch geometry as shown in Fig. 4. Each patch corresponds to a different parametrization, in this case the parametrizations are four different translations. The discretization spaces on each patch are of TH-type with parameters  $d_V = 3, d_p = 2, s_V = s_p = 1$ . The spaces on the multipatch domain are obtained by gluing the patch-local spaces using the technique proposed by Buchegger *et al.* (2016). In this way the basis functions are  $C^1$  on the whole domain.

The error is estimated using the standard residual-based estimator, but due to the  $C^1$  continuity of the basis function the jump terms are 0 and can be ignored. Elements are marked according to Dörfler's technique with parameter  $1/5$ , i.e., the error estimates of the refined elements sum up to a  $1/5$  of the total estimated error.

The mesh was correctly refined near the singularities as can be observed in Fig. 5. The convergence of the error estimator for both uniform and adaptive refinement is reported in Fig 7. Uniform refinement achieves the predicted order of convergence (0.27...) while adaptive refinement achieves an order of 0.94... on average and in line with a theoretical convergence rate of  $\frac{3}{2}$  in the last two iterations. Fig. 6 shows the magnitude of the velocity.



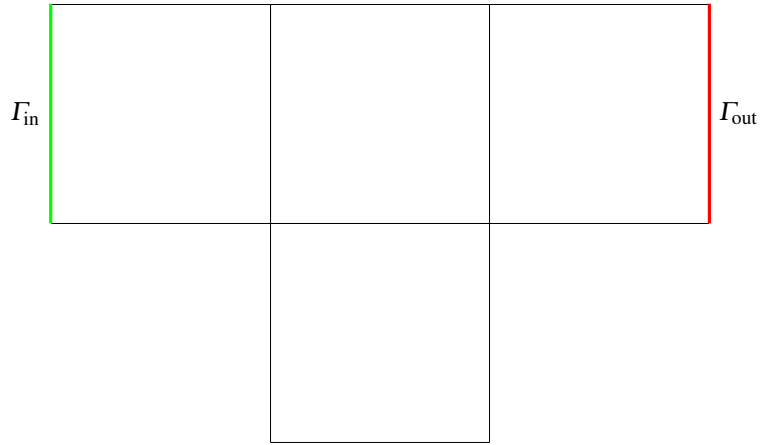


Figure 4. The domain and its subdivision in 4 patches.

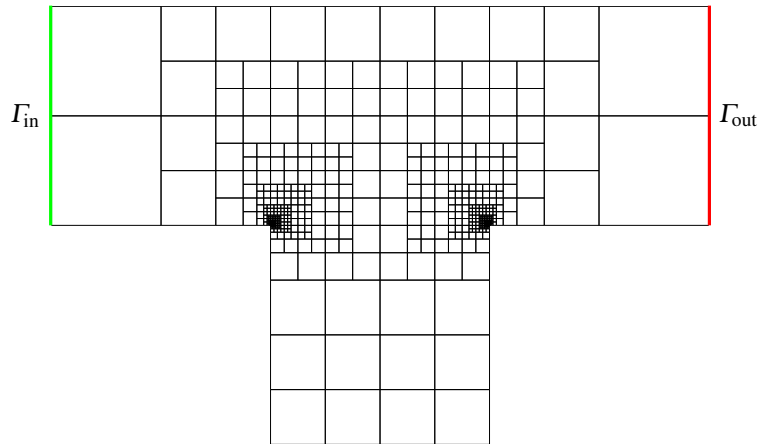


Figure 5. Mesh after 10 refinements steps.

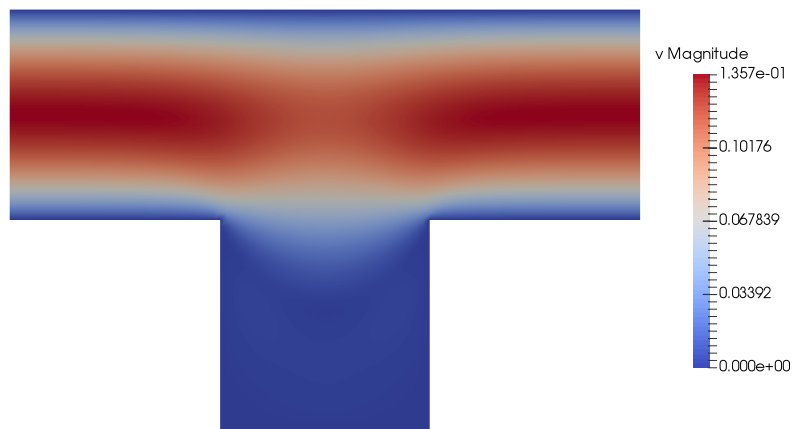


Figure 6. Magnitude of the computed velocity field.

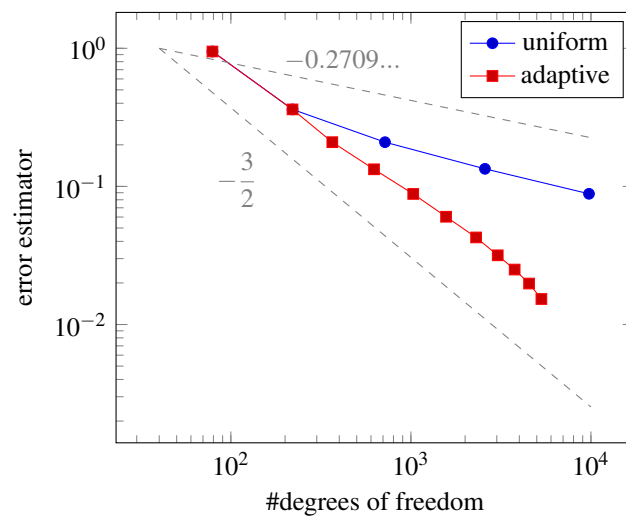


Figure 7. Convergence plot of the adaptive and uniform refining strategy for the problem on the T-shaped domain.

## 6. Summary

This paper extends the stability result of the isogeometric TH- and SG-type methods to hierarchical splines. This is achieved by defining an appropriate approximation operator and then following the same techniques used by Bressan & Sangalli (2013). The inf-sup stability can be achieved for domains of any dimension and the method is thus applicable to both 2D and 3D problems. The convergence of a corresponding adaptive TH scheme was tested numerically on a 2D problem with corner singularities giving good results.

Numerical experiments hint that the lower bound for the discrete inf-sup constant  $\beta_h$  is independent of the degree, mesh-quasi-uniformity and other parameters appearing in the theoretical analysis. From this point of view the theory is not completely satisfactory.

## Acknowledgements

This research was supported from the European Research Council through the following grants: GA 324340-EXAMPLE-FP7-People-2012 IAPP and GA 678727-MOTOR-H2020-FoF-2014-2015/H2020-FoF-2015. This support is gratefully acknowledged.

## References

- ACOSTA, G., DURÁN, R. G. & MUSCHIETTI, M. A. (2006) Solutions of the divergence operator on John domains. *Adv. Math.*, **206**, 373–401.
- BAZILEVS, Y., BEIRÃO DA VEIGA, L., COTTRELL, J. A., HUGHES, T. J. R. & SANGALLI, G. (2006) Isogeometric analysis: approximation, stability and error estimates for  $h$ -refined meshes. *Math. Models Methods Appl. Sci.*, **16**, 1031–1090.
- BEIRÃO DA VEIGA, L., BUFFA, A., SANGALLI, G. & VÁZQUEZ, R. (2014) Mathematical analysis of variational isogeometric methods. *Acta Numer.*, **23**, 157–287.
- BERDINSKY, D., KIM, T.-W., BRACCO, C., CHO, D., MOURRAIN, B., OH, M.-J. & KIATPANICHGIJ, S. (2014) Dimensions and bases of hierarchical tensor-product splines. *J. Comput. Appl. Math.*, **257**, 86–104.
- BORNEMANN, P. B. & CIRAK, F. (2013) A subdivision-based implementation of the hierarchical b-spline finite element method. *Comput. Methods Appl. Mech. Engrg.*, **253**, 584–598.
- BRESSAN, A. (2011) Isogeometric regular discretization for the Stokes problem. *IMA J. Numer. Anal.*, **31**, 1334–1356.
- BRESSAN, A. & SANGALLI, G. (2013) Isogeometric discretizations of the Stokes problem: stability analysis by the macroelement technique. *IMA J. Numer. Anal.*, **33**, 629–651.
- BREZZI, F. & FORTIN, M. (1991) *Mixed and hybrid finite element methods*. Springer Series in Computational Mathematics, vol. 15. Springer-Verlag, New York, pp. x+350.
- BUCHEGGER, F., JÜTTLER, B. & MANTZAFLARIS, A. (2016) Adaptively refined multi-patch B-splines with enhanced smoothness. *Applied Mathematics and Computation*, **272**, 159–172.
- BUFFA, A., RIVAS, J., SANGALLI, G. & VÁZQUEZ, R. (2011) Isogeometric discrete differential forms in three dimensions. *SIAM J. Numer. Anal.*, **49**, 818–844.

- BUFFA, A., SANGALLI, G. & VÁZQUEZ, R. (2014) Isogeometric methods for computational electromagnetics: B-spline and T-spline discretizations. *J. Comput. Phys.*, **257**, 1291–1320.
- BUFFA, A., GIANNELLI, C., MORGENSTERN, P. & PETERSEIM, D. (2016) Complexity of hierarchical refinement for a class of admissible mesh configurations. *Comput. Aided Geom. Design*, **47**, 83–92.
- BUFFA, A. & GIANNELLI, C. (2016) Adaptive isogeometric methods with hierarchical splines: error estimator and convergence. *Math. Models Methods Appl. Sci.*, **26**, 1–25.
- COSTABEL, M., CROUZEIX, M., DAUGE, M. & LAFRANCHE, Y. (2015) The inf-sup constant for the divergence on corner domains. *Numer. Methods Partial Differential Equations*, **31**, 439–458.
- COSTABEL, M. & DAUGE, M. (2015) On the Inequalities of Babuška–Aziz, Friedrichs and Horgan–Payne. *Arch. Ration. Mech. Anal.*, **217**, 873–898.
- COTTRELL, J. A., HUGHES, T. J. R. & BAZILEVS, Y. (2009) *Isogeometric Analysis: Toward Integration of CAD and FEA*. Chichester, England: John Wiley & sons.
- DAUGE, M. (1989) Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I. Linearized equations. *SIAM J. Math. Anal.*, **20**, 74–97.
- ENGELMAN, M. & BERCOVIER, M. (1979) Estimation de l’erreur pour la résolution du problème de Stokes par un élément fini quadrilatéral conforme sur les vitesses. *C. R. Acad. Sci. Paris Sér. A-B*, **288**, A555–A557.
- EVANS, J. A., SCOTT, M. A., THOMAS, D. & VÁZQUEZ, R. (2015) Isogeometric discrete differential forms with hierarchical splines. *Book of Abstract of the III International Conference on Isogeometric Analysis–IGA 2015* (T. Kvamsdal, T. Dokken & K. M. Mathisen eds). CIMNE, p. 33.
- EVANS, J. A. & HUGHES, T. J. R. (2013a) Isogeometric divergence-conforming B-splines for the Darcy-Stokes-Brinkman equations. *Math. Models Methods Appl. Sci.*, **23**, 671–741.
- EVANS, J. A. & HUGHES, T. J. R. (2013b) Isogeometric divergence-conforming B-splines for the steady Navier-Stokes equations. *Math. Models Methods Appl. Sci.*, **23**, 1421–1478.
- FORSEY, D. R. & BARTELS, R. H. (1988) Hierarchical B-spline refinement. *SIGGRAPH Comput. Graph.*, **22**, 205–212.
- GIANNELLI, C., JÜTTLER, B. & SPELEERS, H. (2012) THB-splines: the truncated basis for hierarchical splines. *Comput. Aided Geom. Design*, **29**, 485–498.
- GIANNELLI, C., JÜTTLER, B. & SPELEERS, H. (2014) Strongly stable bases for adaptively refined multilevel spline spaces. *Adv. Comput. Math.*, **40**, 459–490.
- GIANNELLI, C. & JÜTTLER, B. (2013a) Bases and dimensions of bivariate hierarchical tensor-product splines. *J. Comput. Appl. Math.*, **239**, 162–178.
- GIANNELLI, C. & JÜTTLER, B. (2013b) Local and adaptive refinement with hierarchical B-splines. *Boll. Unione Mat. Ital. (9)*, **6**, 735–740.

- HUGHES, T. J. R., COTTRELL, J. A. & BAZILEVS, Y. (2005) Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, **194**, 4135–4195.
- JOHANNESSEN, K. A., KUMAR, M. & KVAMSDAL, T. (2015) Divergence-conforming discretization for Stokes problem on locally refined meshes using LR B-splines. *Comput. Methods Appl. Mech. Engrg.*, **293**, 38–70.
- KURU, G., VERHOOSEL, C. V., VAN DER ZEE, K. G. & VAN BRUMMELEN, E. H. (2014) Goal-adaptive isogeometric analysis with hierarchical splines. *Comput. Methods Appl. Mech. Engrg.*, **270**, 270–292.
- MOKRIŠ, D., JÜTTLER, B. & GIANNELLI, C. (2014) On the completeness of hierarchical tensor-product B-splines. *J. Comput. Appl. Math.*, **271**, 53–70.
- RÜBERG, T. & CIRAK, F. (2012) Subdivision-stabilised immersed b-spline finite elements for moving boundary flows. *Comput. Methods Appl. Mech. Engrg.*, **209/212**, 266–283.
- SPELEERS, H. & MANNI, C. (2016) Effortless quasi-interpolation in hierarchical spaces. *Numer. Math.*, **132**, 155–184.
- STENBERG, R. (1990) Error analysis of some finite element methods for the Stokes problem. *Math. Comp.*, **54**, 495–508.
- VERFÜRTH, R. (1984) Error estimates for a mixed finite element approximation of the Stokes equations. *RAIRO Anal. Numér.*, **18**, 175–182.
- VUONG, A.-V., GIANNELLI, C., JÜTTLER, B. & SIMEON, B. (2011) A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, **200**, 3554–3567.