# An osculating motion with second order contact for spatial Euclidean motions

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**Abstract.** A unique symmetric gliding double screw (SGDS–) motion is constructed which possesses a second order contact with an arbitrary given Euclidean motion at a given position. Additionally it is shown that generally no double screw motion exists which has this property. The constructed SGDS–motion can be said to be the analogue of the osculating circle of a curve for spatial Euclidean motions.

## Introduction

We discuss the construction of an osculating motion with second order contact for spatial Euclidean motions. The obtained osculating motion can be said to be a kinematical analogue of the osculating circle of a curve. This circle serves as a visualization of the curvature properties of a curve in the neighbourhood of one of its points. It became very popular in differential geometry, but also in various applications of Computer Aided Geometric Design, especially in curve and surface interrogation, see e.g. [6, 9]. In contrast with this, most textbooks on spatial kinematics discuss the local differential-geometrical properties of a motion with the help of Taylor expansions, e.g., of the matrix representation of a motion, as introduced by Veldkamp [2, 13]. For more information on recent developments in kinematical geometry the reader is referred to the excellent survey article by Pottmann [10].

The analytical method in spatial kinematics has proved to be very powerful, but a kinematical analogue of the osculating circle can help to develop a better qualitative understanding of the occurring phenomena. Note that the instantaneous screw motion can be said to be the kinematical analogue of the tangent of a curve.

At first we briefly summarize some fundamentals of spatial kinematics and introduce some notations. After discussing spherical motions in Section 2, we examine the local approximation of spatial motions by double screw motions (Section 3) and by symmetric gliding double screw (SGDS–) motions (Section 4). Finally, some first and second order properties of SGDS–motions will be outlined.

## 1. Preliminaries

Consider two coinciding three–dimensional Euclidean spaces, the *fixed space* E and the *moving space* $\widehat{\mathrm{E}}$, both equipped with Cartesian frames $(\mathbf{O};\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ and $(\hat{\mathbf{O}};\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3)$, respectively. The points of both spaces will be identified with their coordinate vectors $\mathbf{a},\mathbf{b},\ldots$ resp. $\hat{\mathbf{a}},\hat{\mathbf{b}},\ldots$ from $\mathbb{R}^3$. A Euclidean one–parameter motion of the moving space $\widehat{\mathrm{E}}$ with respect to the fixed space E is assumed to be given. This motion is described by the one parameter set of mappings

$$\begin{array}{rlcl} \mathrm{M}=\mathrm{M}(t): & \widehat{\mathrm{E}}\times\mathbb{R} & \to & \mathrm{E} \\ & \hat{\mathbf{x}} & \mapsto & \mathbf{x}=\vec{\mathbf{b}}(t)+\mathrm{U}(t)\hat{\mathbf{x}} \end{array} \qquad (1)$$

whereby the parameter $t\in\mathbb{R}$ may be identified with the time. At any time $t$, the moving space results from the fixed space by the spatial displacement $\mathrm{M}(t)$. This displacement is obtained by composing a translation by the vector $\vec{\mathbf{b}}=\vec{\mathbf{b}}(t)$ and a rotation described by the orthonormal real $3\times 3$–matrix $\mathrm{U}=\mathrm{U}(t)$.

The vector $\vec{\mathbf{b}}(t)$ and the matrix $\mathrm{U}(t)$ are presumed to be at least twice continuously differentiable with respect to the time $t$. We will discuss the first and second order properties of the motion $\mathrm{M}(t)$ in the neighbourhood of a fixed time $t_0\in\mathbb{R}$. Without loss of generality, the spatial displacement $\mathrm{M}(t_0)$ is assumed to be the identical mapping $(\vec{\mathbf{b}}(t_0)=\vec{\mathbf{0}},\mathrm{U}(t_0)=\mathrm{I})$. This can always be achieved by choosing an appropriate coordinate frame in the moving space.

Consider an arbitrary point $\hat{\mathbf{x}}\in\widehat{\mathrm{E}}$ of the moving space. Its velocity and acceleration are

$$\begin{array}{rcl} \dot{\vec{\mathbf{b}}}(t)+\dot{\mathrm{U}}(t)\,\hat{\mathbf{x}} & = & \dot{\vec{\mathbf{b}}}(t)+\vec{\mathbf{r}}(t)\times\hat{\mathbf{x}} \quad \text{and} \\ \ddot{\vec{\mathbf{b}}}(t)+\ddot{\mathrm{U}}(t)\,\hat{\mathbf{x}} & = & \ddot{\vec{\mathbf{b}}}(t)+\dot{\vec{\mathbf{r}}}(t)\times\hat{\mathbf{x}} \quad , \end{array} \qquad (2)$$

whereby the dot "˙" denotes the differentiation with respect to $t$. Note that the products $\dot{\mathrm{U}}(t)\,\hat{\mathbf{x}}$ and $\ddot{\mathrm{U}}(t)\,\hat{\mathbf{x}}$ can be represented by the vector products $\vec{\mathbf{r}}(t)\times\hat{\mathbf{x}}$ and $\dot{\vec{\mathbf{r}}}(t)\times\hat{\mathbf{x}}$, respectively, because differentiating an orthonormal matrix leads to a skew–symmetric matrix, see e.g. [2]. The vector $\vec{\mathbf{r}}=\vec{\mathbf{r}}(t)$ is the angular velocity of the moving space.

The angular velocity vector $\vec{\mathbf{r}}=\vec{\mathbf{r}}(t)$ is assumed to satisfy $\vec{\mathbf{r}}(t)\neq\vec{\mathbf{0}}$ in a neighbourhood of $t=t_0$, i.e., points with vanishing angular velocity have to be excluded in the following considerations.

In order to simplify the calculations, we introduce the canonical parameter $s$,

$$s = s(t) = \int_{t_0}^{t} \|\vec{\mathbf{r}}(t)\| \, \mathrm{d}t \tag{3}$$

whereby $\|\vec{\mathbf{x}}\| = \sqrt{\vec{\mathbf{x}}^\top \vec{\mathbf{x}}}$ denotes the norm of a vector $\vec{\mathbf{x}} \in \mathbb{R}^3$. The first and second order properties of the given motion $\mathrm{M}(t)$ at the time $t = t_0$ will be discussed using the canonical parametrization. These properties are governed by the four vectors

$$\begin{aligned}
\vec{\mathbf{v}} &= \left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{s=0} \dot{\vec{\mathbf{b}}}(t_0) \,, \\
\vec{\mathbf{a}} &= \left.\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\right|_{s=0} \ddot{\vec{\mathbf{b}}}(t_0) + \left.\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}\right|_{s=0} \dot{\vec{\mathbf{b}}}(t_0) \,, \\
\vec{\omega} &= \left.\frac{\mathrm{d}t}{\mathrm{d}s}\right|_{s=0} \vec{\mathbf{r}}(t_0) \quad \text{and} \\
\vec{\alpha} &= \left.\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\right|_{s=0} \dot{\vec{\mathbf{r}}}(t_0) + \left.\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}\right|_{s=0} \vec{\mathbf{r}}(t_0) \,,
\end{aligned} \tag{4}$$

which represent the velocity of the origin, the acceleration of the origin, the angular velocity and the angular acceleration of the moving space with respect to the canonical parameter $s$ at $s = 0$ (i.e., at $t = t_0$), respectively. Due to the definition (3), the vectors of the angular velocity and acceleration satisfy

$$\|\vec{\omega}\| = 1 \quad \text{and} \quad (\vec{\omega}, \vec{\alpha}) = 0 \,, \tag{5}$$

whereby $(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \vec{\mathbf{x}}^\top \vec{\mathbf{y}}$ denotes the inner product of the vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$.

In differential geometry, the osculating circle of a curve at a point is constructed as the unique circle which possesses a *second order contact* with the curve. We will construct a unique spatial motion $\mathrm{C} = \mathrm{C}(s) : \widehat{\mathrm{E}} \times \mathbb{R} \to \mathrm{E}$ which may serve as a kinematical analogue. We define:

**Definition.** Two spatial motions $\mathrm{M}(t(s))$ and $\mathrm{C} = \mathrm{C}(s)$ are said to have a *second order contact* at $s = 0$, if the trajectories $\mathrm{M}(t(s))\,\hat{\mathbf{x}}$ and $\mathrm{C}(s)\,\hat{\mathbf{x}}$ of any point $\hat{\mathbf{x}}$ of the moving space $\widehat{\mathrm{E}}$ have the same Frenet frames and coinciding osculating circles at $s = 0$. This assertion holds if and only if the velocities of the origin, the accelerations of the origin, the angular velocities and the angular accelerations of the moving space with respect to the canonical parameter $s$ at $s = 0$ of both motions are identical.

We will construct the osculating motion $\mathrm{C}(s)$ by composing several elementary motions. By the abbreviations $\mathrm{Trans}(\vec{\mathbf{v}})$ and $\mathrm{Rot}(\vec{\mathbf{r}}, \varphi)$ we denote the translation by the vector $\vec{\mathbf{v}} \in \mathbb{R}^3$ and the rotation (with a right–handed orientation) through the angle $\varphi$ around the axis with the normalized direction vector $\vec{\mathbf{r}} \in \mathbb{R}^3$ ($\|\vec{\mathbf{r}}\| = 1$), respectively. The circle $\circ$ stands for the composition of elementary motions. For instance, any motion $\mathrm{M}(t)$ can be composed from a translational and a rotational part,

$$\mathrm{M}(t) = \mathrm{Trans}(\vec{\mathbf{v}}(t)) \circ \mathrm{Rot}(\vec{\mathbf{r}}(t), \varphi(t)), \tag{6}$$

where $\vec{\mathbf{v}}(t)$, $\vec{\mathbf{r}}(t)$ and $\varphi(t)$ depend on $t$. The rotational motion defined by $\mathrm{Rot}(\vec{\mathbf{r}}(t), \varphi(t))$ is often called the *spherical part* of the given Euclidean motion because it describes a motion on the unit sphere.

Note that in the special case of planar motions, the construction of a kinematical analogue of the osculating circle turns out to be nearly obvious. Resulting from the Euler–Savary formula (see [1, p.29, p.39]), the osculating circles of the centrodes can be replaced by two symmetric circles without influencing the second order properties of the planar motion because only $\left(\frac{1}{r}-\frac{1}{\hat{r}}\right)$ is essential in this formula. (The constants $r$ and $\hat{r}$ denote the radii of the osculating circles of the fixed and of the moving centrode.) Therefore, a unique *symmetric* planar double–rotational motion always exists which possesses a second order contact with the given planar motion at $s = 0$.

## 2. Symmetric double–rotational motions

At first we consider only the rotational (also called spherical) part of the given motion $\mathrm{M}(t)$. The construction of the osculating motion will then turn out to be completely analogous to the planar case.

We examine the spherical motion obtained by composing two uniform rotations with identical absolute values of the angular velocities,

$$\begin{gathered} \mathrm{C}_{\mathrm{rot}}(s) = \mathrm{Rot}(\vec{\mathbf{r}}, \lambda s) \circ \mathrm{Rot}(\vec{\mathbf{q}}, \lambda s) \\ (\vec{\mathbf{r}}, \vec{\mathbf{q}} \in \mathbb{R}^3;\ \|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1;\ \lambda = \frac{1}{\|\vec{\mathbf{r}} + \vec{\mathbf{q}}\|}). \end{gathered} \tag{7}$$

Its centrodes are two symmetric spherical circles. The motion (7) will be called a *symmetric spherical double–rotational motion*. A short calculation leads to the angular velocity and acceleration of the moving space at $s = 0$,

$$\vec{\omega}^{(\mathrm{C})} = \lambda\ (\vec{\mathbf{r}} + \vec{\mathbf{q}}) \quad \text{and} \quad \vec{\alpha}^{(\mathrm{C})} = \lambda^2\ (\vec{\mathbf{r}} \times \vec{\mathbf{q}}), \tag{8}$$

respectively, where $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ denotes the usual vector product of two vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$. Due to the choice of $\lambda$ (see (7)), the parameter $s$ is the canonical parameter of the motion $\mathrm{C}_{\mathrm{rot}}(s)$, because both equations $\|\vec{\omega}^{(\mathrm{C})}\| = 1$ and $(\vec{\alpha}^{(\mathrm{C})}, \vec{\omega}^{(\mathrm{C})}) = 0$ are fulfilled.

Similar to the planar case we have:

**Proposition 1.** *There exists a unique symmetric spherical double–rotational motion (7) which possesses a contact of second order with the spherical part of the given motion* $\mathrm{M}(t(s))$ *at* $s = 0$.

**Proof.** The spherical double–rotational motion $\mathrm{C}_{\mathrm{rot}}(s)$ has a second order contact with the spherical part of $\mathrm{M}(t(s))$ if and only if the two equations

$$\vec{\omega} = \lambda\,(\vec{\mathbf{r}} + \vec{\mathbf{q}}) \tag{9}$$

$$\vec{\alpha} = \lambda^2\,(\vec{\mathbf{r}} \times \vec{\mathbf{q}}) \tag{10}$$

$$\text{with} \quad \lambda = \frac{1}{\|\vec{\mathbf{r}} + \vec{\mathbf{q}}\|} \quad \text{and} \quad \|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1 \tag{11}$$

are fulfilled, where $\vec{\omega}$ and $\vec{\alpha}$ are the angular velocity and acceleration of $\mathrm{M}(t(s))$ at $s = 0$, see (4).

**First case:** $\vec{\alpha} \neq \vec{\mathbf{0}}$. Due to (10), the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ have to be perpendicular to $\vec{\alpha}$. Combining $\|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\|$ with (9) and (10) we obtain

$$\vec{\mathbf{r}} = \frac{1}{\lambda}\left(\frac{1}{2}\vec{\omega} + \vec{\omega} \times \vec{\alpha}\right) \quad \text{and} \quad \vec{\mathbf{q}} = \frac{1}{\lambda}\left(\frac{1}{2}\vec{\omega} - \vec{\omega} \times \vec{\alpha}\right), \tag{12}$$

cf. Figure 1. The conditions $\|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1$ then imply

$$\lambda = \sqrt{\frac{1}{4} + \|\vec{\alpha}\|^2}. \tag{13}$$

Note that the equation (7) for $\lambda$ is fulfilled.

Figure 1: The construction of the spherical double–rotational motion.

**Second case:** $\vec{\alpha} = \vec{\mathbf{0}}$. Resulting from (10) the vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ are linearly dependent. The conditions $\|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1$ imply

$$\vec{\mathbf{r}} = \vec{\mathbf{q}} = \frac{1}{2}\vec{\omega} \tag{14}$$

because the angular velocity $\vec{\omega}$ was assumed to be not the null vector.

In both cases, the direction vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ exist and they are uniquely determined by the given angular velocity and angular acceleration. ■

If the angular acceleration $\vec{\alpha}$ satisfies $\vec{\alpha} \neq \vec{\mathbf{0}}$, then the direction vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ of the axes of the rotations are linearly independent. This will turn out to be necessary for the construction of a symmetric gliding double screw motion having a second order contact with $\mathrm{M}(t(s))$ at $s = 0$, see Section 4.

Note that the above proposition can also be proved geometrically using the spherical formulation of the Euler–Savary formula, cf. [8, p. 16].

## 3. Double screw motions

In order to construct an osculating spatial motion $\mathrm{C}(s)$, we seek for appropriate extensions of the previously constructed spherical motion. In this section, the *double screw motion*

$$\begin{aligned}\mathrm{C}^{(\mathrm{DS})}(s) = \; & \mathrm{Trans}(\vec{\mathbf{x}}) \circ \underbrace{\mathrm{Rot}(\vec{\mathbf{r}}, \lambda s) \circ \mathrm{Trans}(s\,\mu\,\vec{\mathbf{r}})}_{\text{first screw motion}} \circ \mathrm{Trans}(-\vec{\mathbf{x}}) \\ & \circ \mathrm{Trans}(\vec{\mathbf{y}}) \circ \underbrace{\mathrm{Rot}(\vec{\mathbf{q}}, \lambda s) \circ \mathrm{Trans}(s\,\nu\,\vec{\mathbf{q}})}_{\text{second screw motion}} \circ \mathrm{Trans}(-\vec{\mathbf{y}})\end{aligned} \tag{15}$$

(see Figure 2) will be examined. This motion is defined by the two axes $\vec{\mathbf{x}} + \tau\vec{\mathbf{r}}$ and $\vec{\mathbf{y}} + \tau\vec{\mathbf{q}}$ ($\tau \in \mathbb{R}$). The moving space $\widehat{\mathrm{E}}$ is subject to a screw motion around the second axis, whereby the second axis simultaneously performs a screw motion around the first axis. The axodes of the double screw motion (15) are two ruled helical surfaces. A detailed discussion of such motions (also called double–helical motions) can be found in the textbook [7] and in the article by Hohenberg [5]. The latter article also provides a beautiful figure of the resulting point trajectories.

Figure 2: The double screw motion (scheme).

The direction vectors $\vec{\mathbf{r}}, \vec{\mathbf{q}} \in \mathbb{R}^3$ ($\|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1$) of the screw axes and the constant $\lambda \in \mathbb{R}$ are taken from the osculating symmetric spherical double–rotational motion (7), see Proposition 1. The spherical part of $\mathrm{C}^{(\mathrm{DS})}(s)$ therefore possesses a second order contact with that of $\mathrm{M}(t(s))$ at $s = 0$. Resulting from the choice of $\lambda$, the parameter $s$ is again the canonical parameter of the motion $\mathrm{C}^{(\mathrm{DS})}(s)$.

The translation vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$ (which determine the position of the screw axes) and the screw parameters (the pitches) $\mu, \nu \in \mathbb{R}$ are unknown yet. Without loss of generality,

$$(\vec{\mathbf{x}}, \vec{\mathbf{r}}) = (\vec{\mathbf{y}}, \vec{\mathbf{q}}) = 0 \tag{16}$$

will be assumed. Some calculations lead to the velocity $\vec{\mathbf{v}}^{(\mathrm{DS})}$ and the acceleration $\vec{\mathbf{a}}^{(\mathrm{DS})}$ of the origin of the moving space under the double screw motion (15) at $s = 0$:

$$\begin{aligned}\vec{\mathbf{v}}^{(\mathrm{DS})} &= \underbrace{\vec{\mathbf{x}} \times (\lambda\,\vec{\mathbf{r}}) + \mu\,\vec{\mathbf{r}}}_{\text{first screw motion}} + \underbrace{\vec{\mathbf{y}} \times (\lambda\,\vec{\mathbf{q}}) + \nu\,\vec{\mathbf{q}}}_{\text{second screw motion}} \qquad \text{and} \\ \vec{\mathbf{a}}^{(\mathrm{DS})} &= \underbrace{\lambda^2\,\vec{\mathbf{r}} \times (\vec{\mathbf{x}} \times \vec{\mathbf{r}})}_{(i)} + \underbrace{\lambda^2\,\vec{\mathbf{q}} \times (\vec{\mathbf{y}} \times \vec{\mathbf{q}})}_{(ii)} + \underbrace{2\,\lambda\,\vec{\mathbf{r}} \times [\,\nu\,\vec{\mathbf{q}} + \vec{\mathbf{y}} \times (\lambda\,\vec{\mathbf{q}})\,]}_{(iii)}\,.\end{aligned} \tag{17}$$

The first two terms $(i)$ and $(ii)$ result from the rotations around the screw axes, whereas $(iii)$ is the Coriolis acceleration generated by the velocity of the second screw motion. The double screw motion (15) has a second order contact with $\mathrm{M}(t(s))$ at $s=0$ if and only if the two conditions $\vec{\mathbf{v}}^{(\mathrm{DS})}=\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}^{(\mathrm{DS})}=\vec{\mathbf{a}}$ are fulfilled.

In order to compute the unknown translation vectors $\vec{\mathbf{x}},\vec{\mathbf{y}}\in\mathbb{R}^3$ and screw parameters $\mu,\nu\in\mathbb{R}$, a special Cartesian coordinate system will be used. The direction of the $x$–axis is assumed to be that of the first screw axis $\vec{\mathbf{r}}$, and the $xy$–plane should be parallel to $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$. Under these assumptions we have

$$\vec{\mathbf{r}}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \vec{\mathbf{q}}=\begin{pmatrix}q_1\\q_2\\0\end{pmatrix}\quad (q_1,q_2\in\mathbb{R};\ \mathrm{q}_1^2+\mathrm{q}_2^2=1)\quad\text{and}\quad \lambda=\frac{1}{\sqrt{(1+\mathrm{q}_1)^2+\mathrm{q}_2^2}}\,. \tag{18}$$

Additionally, the abbreviations

$$\vec{\mathbf{x}}\times(\lambda\,\vec{\mathbf{r}})=\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\quad\text{and}\quad \vec{\mathbf{y}}\times(\lambda\,\vec{\mathbf{q}})=\begin{pmatrix}-b\,q_2\\bq_1\\y_3\end{pmatrix} \tag{19}$$

$(x_2,x_3,b,y_3\in\mathbb{R})$ will be used. Note that if the vectors $\vec{\mathbf{x}}\times(\lambda\,\vec{\mathbf{r}})$ and $\vec{\mathbf{y}}\times(\lambda\,\vec{\mathbf{q}})$ are known, then $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ uniquely result from the conditions (16),

From (17) and from the velocity $\vec{\mathbf{v}}$ and acceleration $\vec{\mathbf{a}}$ of the origin under the given motion $\mathrm{M}(t(s))$ we obtain a system of six linear equations for the six unknown parameters $x_2,x_3,b,y_3,\mu$ and $\nu$ of the double–screw motion (15). The remaining two parameters $q_1,q_2$ are already known from the rotational part. Surprisingly one gets:

**Theorem 2.** *Generally, no double screw motion (15) exists which possesses a second order contact with the given motion* $\mathrm{M}(t(s))$ *at* $s=0$.

**Proof.** From (17) and (19) we obtain the six linear equations

$$\begin{aligned}\vec{\mathbf{v}}&=\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}\mu\\0\\0\end{pmatrix}+\begin{pmatrix}-b\,q_2\\bq_1\\y_3\end{pmatrix}+\begin{pmatrix}\nu\,q_1\\\nu\,q_2\\0\end{pmatrix}\\ \vec{\mathbf{a}}&=\begin{pmatrix}\lambda\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}\lambda\,q_1\\\lambda\,q_2\\0\end{pmatrix}\times\begin{pmatrix}-b\,q_2\\bq_1\\y_3\end{pmatrix}+\begin{pmatrix}2\,\lambda\\0\\0\end{pmatrix}\times\left\{\begin{pmatrix}\nu\,q_1\\\nu\,q_2\\0\end{pmatrix}+\begin{pmatrix}-b\,q_2\\bq_1\\y_3\end{pmatrix}\right\}\end{aligned} \tag{20}$$

for the six unknown parameters $x_2,x_3,b,y_3,\mu$ and $\nu$. The third, fourth, and fifth equation of this system are

$$\begin{aligned}v_3&=&x_3&+&y_3\\ a_1&=&&+&\lambda\,q_2\,y_3\\ a_2&=&-\lambda\,x_3&+&(-\lambda q_1-2\,\lambda)y_3\end{aligned} \tag{21}$$

where $\vec{\mathbf{v}}=(v_1\ v_2\ v_3)^\top$ and $\vec{\mathbf{a}}=(a_1\ a_2\ a_3)^\top$. They form a system of three linear equations for only two unknown parameters $x_3$ and $y_3$, therefore generally no solutions exist. ■

Later, a geometrical approach to the theorem will be outlined, see Section 5. This result on double screw motions is quite surprising, because the number of parameters in the system (20) is equal to the number of equations. Moreover, a spatial formulation of the Euler–Savary formula exists [2, 4, 12]. This formula deals with the Disteli axes (also called instantaneous screw axes) of the ruled surfaces generated by a moving line (trajectory ruled surfaces). But the geometry of spatial motions is more complicated than that of planar or of spherical motions.

## 4. Symmetric gliding double screw motions

Now we consider the motion

$$\begin{aligned} \mathrm{C}^{(\mathrm{SGDS})}(s) = {} & \mathrm{Trans}(\vec{\mathbf{x}}) \circ \underbrace{\mathrm{Rot}(\vec{\mathbf{r}}, \lambda s) \circ \mathrm{Trans}(s\,\mu\,\vec{\mathbf{r}})}_{\text{first screw motion}} \circ \mathrm{Trans}(-\vec{\mathbf{x}}) \\ & \circ \underbrace{\mathrm{Trans}(s\,\varrho\,\vec{\mathbf{r}} \times \vec{\mathbf{q}})}_{\text{translational motion}} \\ & \circ \mathrm{Trans}(\vec{\mathbf{y}}) \circ \underbrace{\mathrm{Rot}(\vec{\mathbf{q}}, \lambda s) \circ \mathrm{Trans}(s\,\mu\,\vec{\mathbf{q}})}_{\text{second screw motion}} \circ \mathrm{Trans}(-\vec{\mathbf{y}}) \end{aligned} \tag{22}$$

(see Figure 3) which results from (15) by introducing an additional translational motion with the velocity vector $\varrho\,\vec{\mathbf{r}} \times \vec{\mathbf{q}}$ ($\varrho \in \mathbb{R}$) (i.e., the velocity is perpendicular to both screw axes) between the screw motions. Additionally, the screw parameters $\mu$ and $\nu$ are assumed to be identical.

Figure 3: The symmetric gliding double screw motion (scheme).

The motion (22) will be called a *symmetric gliding double screw motion* (SGDS–motion). Again, the direction vectors $\vec{\mathbf{r}}, \vec{\mathbf{q}} \in \mathbb{R}^3$ ($\|\vec{\mathbf{r}}\| = \|\vec{\mathbf{q}}\| = 1$) of the screw axes and the constant $\lambda \in \mathbb{R}$ are taken from the symmetric double–rotational motion (7) of Proposition 1. The rotational part of $\mathrm{C}^{(\mathrm{SGDS})}(s)$ therefore has a second order contact with that of $\mathrm{M}(t(s))$ at $s = 0$. Moreover, the parameter $s$ is the canonical parameter of the SGDS–motion (22).

The direction vectors $\vec{\mathbf{r}}, \vec{\mathbf{q}}$ of the screw axes are assumed to be linearly independent. The translation vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$ and the parameters $\mu, \varrho \in \mathbb{R}$ are unknown yet. Again, the vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ are to fulfill the orthogonality conditions (16).

Some calculations lead to the expressions

$$\begin{aligned} \vec{\mathbf{v}}^{(\mathrm{SGDS})} &= \underbrace{\vec{\mathbf{x}} \times (\lambda\,\vec{\mathbf{r}}) + \mu\,\vec{\mathbf{r}}}_{\text{first screw motion}} + \underbrace{\varrho\,(\vec{\mathbf{r}} \times \vec{\mathbf{q}})}_{\text{translational motion}} + \underbrace{\vec{\mathbf{y}} \times (\lambda\,\vec{\mathbf{q}}) + \mu\,\vec{\mathbf{q}}}_{\text{second screw motion}} \quad \text{and} \\ \vec{\mathbf{a}}^{(\mathrm{SGDS})} &= \underbrace{\lambda^2\,\vec{\mathbf{r}} \times (\vec{\mathbf{x}} \times \vec{\mathbf{r}})}_{(i)} + \underbrace{\lambda^2\,\vec{\mathbf{q}} \times (\vec{\mathbf{y}} \times \vec{\mathbf{q}})}_{(ii)} + \underbrace{2\,\lambda\,\vec{\mathbf{r}} \times [\,\varrho\,(\vec{\mathbf{r}} \times \vec{\mathbf{q}}) + \mu\,\vec{\mathbf{q}} + \vec{\mathbf{y}} \times (\lambda\,\vec{\mathbf{q}})\,]}_{(iii)} \end{aligned} \tag{23}$$

for the velocity $\vec{\mathbf{v}}^{(\mathrm{SGDS})}$ and the acceleration $\vec{\mathbf{a}}^{(\mathrm{SGDS})}$ of the origin of the moving space under the SGDS–motion (15) at $s = 0$. The terms $(i)$–$(iii)$ result similarly to those of Eq. (17). The SGDS–motion (22) has a second order contact with $\mathrm{M}(t(s))$ at $s = 0$ if and only if the two conditions $\vec{\mathbf{v}}^{(\mathrm{SGDS})} = \vec{\mathbf{v}}$ and $\vec{\mathbf{a}}^{(\mathrm{SGDS})} = \vec{\mathbf{a}}$ are fulfilled. In order to compute the translation vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$ and the parameters $\mu, \varrho \in \mathbb{R}$, the same coordinate system as introduced in the previous section (see (18) and (19)) will be used and we obtain a system of six linear equations for the six unknown parameters $x_2, x_3, b, y_3, \mu$ and $\varrho$. Examining this system leads to

**Theorem 3.** *If the angular acceleration vector $\vec{\alpha}$ does not vanish ($\vec{\alpha} \neq \vec{\mathbf{0}}$), then a unique symmetric gliding double screw motion (22) exists which possesses a second order contact with $\mathrm{M}(t(s))$ at $s = 0$. This motion is obtained independently of the choice of the coordinate systems of the fixed and of the moving space. Moreover, applying the construction to the inverse motion $\mathrm{M}(t(s))^{-1}$ leads to the inverse SGDS–motion.*

**Proof.** From Equations (19) and (23) we get the six linear equations

$$\vec{\mathbf{v}} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varrho q_2 \end{pmatrix} + \begin{pmatrix} -b\, q_2 \\ b q_1 \\ y_3 \end{pmatrix} + \begin{pmatrix} \mu\, q_1 \\ \mu\, q_2 \\ 0 \end{pmatrix}$$
$$\vec{\mathbf{a}} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \lambda\, q_1 \\ \lambda\, q_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -b\, q_2 \\ b q_1 \\ y_3 \end{pmatrix} + \begin{pmatrix} 2\,\lambda \\ 0 \\ 0 \end{pmatrix} \times \left\{ \begin{pmatrix} \mu\, q_1 - b\, q_2 \\ \mu\, q_2 + b\, q_1 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varrho q_2 \end{pmatrix} \right\} \tag{24}$$

for the six unknown parameters $x_2, x_3, b, y_3, \mu$ and $\varrho$. Computing the determinant of the coefficient matrix of the linear system yields

$$D = -\lambda^3\, q_2^2\, \left[ q_2^2 + (1 + q_1)^2 \right]. \tag{25}$$

Due to the assumption $\vec{\alpha} \neq 0$, the vectors $\vec{\mathbf{r}}, \vec{\mathbf{q}}$ are linearly independent, thus $\lambda \neq 0$ and $q_2 \neq 0$ hold (cf. (7) and (18)). Resulting from $D \neq 0$, the system (24) leads to a unique set of solutions, therefore we obtain a unique osculating SGDS–motion.

Now consider the choice of another coordinate system of the fixed space. For instance, by moving the origin of the fixed space we obtain from (22) the motion

$$\mathrm{Trans}(\vec{\mathbf{z}}) \,\circ\, \mathrm{C}^{(\mathrm{SGDS})}(s) \,\circ\, \mathrm{Trans}(-\vec{\mathbf{z}}) \tag{26}$$

where $\vec{\mathbf{z}} \in \mathbb{R}^3$ is the translation vector to the old origin. Note that the assumptions of Section 1 imply the simultaneous choice of another origin of the moving space because $\mathrm{M}(t(s))$ was assumed to be the identical mapping. Let

$$\vec{\mathbf{z}} = \vec{\mathbf{z}}_{||r} + \vec{\mathbf{z}}_{\perp r} = \vec{\mathbf{z}}_{||q} + \vec{\mathbf{z}}_{\perp q} \tag{27}$$

where $\vec{\mathbf{z}}_{||r}$ ($\vec{\mathbf{z}}_{||q}$) and $\vec{\mathbf{z}}_{\perp r}$ ($\vec{\mathbf{z}}_{\perp q}$) denote the components of $\vec{\mathbf{z}}$ which are parallel and perpendicular with respect to $\vec{\mathbf{r}}$ (to $\vec{\mathbf{q}}$), respectively. By inserting the identical mapping

$$\mathrm{I} = \mathrm{Trans}(-\vec{\mathbf{z}}_{||r} - \vec{\mathbf{z}}_{\perp r}) \ \circ \ \mathrm{Trans}(\vec{\mathbf{z}}_{||q} + \vec{\mathbf{z}}_{\perp q}) \tag{28}$$

between the two screw motions of (22) one obtains after some calculations

$$\begin{aligned}
&\mathrm{Trans}(\vec{\mathbf{z}}) \ \circ \ \mathrm{C}^{(\mathrm{SGDS})}(s) \ \circ \ \mathrm{Trans}(-\vec{\mathbf{z}}) \\
&\quad = \mathrm{Trans}(\vec{\mathbf{z}}_{||r} + \vec{\mathbf{z}}_{\perp r}) \ \circ \ \mathrm{C}^{(\mathrm{SGDS})}(s) \ \circ \ \mathrm{Trans}(-\vec{\mathbf{z}}_{||q} - \vec{\mathbf{z}}_{\perp q}) \\
&\quad = \mathrm{Trans}(\vec{\mathbf{x}} + \vec{\mathbf{z}}_{\perp r}) \ \circ \ \mathrm{Rot}(\vec{\mathbf{r}}, \lambda s) \ \circ \ \mathrm{Trans}(s\,\mu\,\vec{\mathbf{r}}) \ \circ \ \mathrm{Trans}(-\vec{\mathbf{x}} - \vec{\mathbf{z}}_{\perp r}) \\
&\qquad\quad \circ \ \mathrm{Trans}(s\,\varrho\,\vec{\mathbf{r}} \times \vec{\mathbf{q}}) \\
&\qquad\qquad \circ \ \mathrm{Trans}(\vec{\mathbf{y}} + \vec{\mathbf{z}}_{\perp q}) \ \circ \ \mathrm{Rot}(\vec{\mathbf{q}}, \lambda s) \ \circ \ \mathrm{Trans}(s\,\mu\,\vec{\mathbf{q}}) \ \circ \ \mathrm{Trans}(-\vec{\mathbf{y}} - \vec{\mathbf{z}}_{\perp q}),
\end{aligned} \tag{29}$$

hence we get again an SGDS–motion of the form (22). (Note that translations and rotations can be permuted if the translation vector and the axis of the rotation are parallel.) The choice of another orientation of the fixed space (i.e., a rotation of the coordinate systems) and the inversion of the motion can be discussed similarly. ■

*The unique SGDS–motion from Theorem 3 can be said to be a kinematical analogue of the osculating circle.* Based on Theorem 3, the local first and second order properties of any Euclidean motion $\mathrm{M} = \mathrm{M}(t(s))$ satisfying $\vec{\alpha} \neq \vec{\mathbf{0}}$ can be discussed with the help of this motion. For instance, the velocity $\vec{\mathbf{v}}_p$ and the acceleration $\vec{\mathbf{a}}_p$ of an arbitrary point $\hat{\mathbf{p}} \in \widehat{\mathrm{E}}$ immediately result from (23), whereby $\vec{\mathbf{x}}$ resp. $\vec{\mathbf{y}}$ are replaced by the distance vectors from $\hat{\mathbf{p}}$ to the first resp. to the second screw axis. Then, the *axis of curvature* (i.e., the locus of the midpoints of all spheres which have a second order contact with the trajectory of $\hat{\mathbf{p}}$ at $s = 0$) can be obtained from *Stübler's construction* [11]: It is the polar of the line $\hat{\mathbf{p}} + \vec{\mathbf{a}}_p + \tau\vec{\mathbf{v}}_p$ ($\tau \in \mathbb{R}$) with respect to the sphere with centre $\hat{\mathbf{p}}$ and radius $\|\vec{\mathbf{v}}_p\|$. Restricting Stübler's construction to the osculating plane of the trajectory leads to a construction of the midpoint of the osculating circle which is known as Bereis' construction in planar kinematics [8].

## 5. Some properties of SGDS–motions

Consider again an SGDS–motion (22), where the directions $\vec{\mathbf{r}}$ and $\vec{\mathbf{q}}$ of the screw axes are assumed to be linearly independent. In order to discuss some of its properties we consider an arbitrary point

$$\hat{\mathbf{p}} = \hat{\mathbf{p}}_0 + u\,\vec{\mathbf{r}} + v\,\vec{\mathbf{q}} + w\,\vec{\mathbf{r}} \times \vec{\mathbf{q}} \tag{30}$$

($u, v, w \in \mathbb{R}$) of the moving space $\widehat{\mathrm{E}}$. Hereby, $\hat{\mathbf{p}}_0$ means the intersection of the common perpendicular of the screw axes with the first screw axis, cf. Figure 3. The velocity vector $\vec{\mathbf{v}}_p$

of $\hat{\mathbf{p}}$ results from (23),

$$\vec{\mathbf{v}}_p = \underbrace{[-\lambda((\vec{\mathbf{r}},\vec{\mathbf{q}})+1)w - d\,\lambda + \mu]}_{=A_1(w)}\,\vec{\mathbf{r}} + \underbrace{[\lambda((\vec{\mathbf{r}},\vec{\mathbf{q}})+1)w + d(\vec{\mathbf{r}},\vec{\mathbf{q}})\lambda + \mu]}_{=A_2(w)}\,\vec{\mathbf{q}} + \underbrace{[(u-v)\lambda + \varrho]}_{=A_3(u,v)}\,\vec{\mathbf{r}}\times\vec{\mathbf{q}}, \tag{31}$$

where $d$ denotes the oriented distance of the screw axes (i.e., the point $\hat{\mathbf{p}}_0 - d\,\vec{\mathbf{r}}\times\vec{\mathbf{q}}$ lies on the second screw axis).

Resulting from (23), the instantaneous axis of the SGDS–motion at $s = 0$ is parallel to $\vec{\mathbf{r}}+\vec{\mathbf{q}}$. Thus, the point $\hat{\mathbf{p}}$ lies on the instantaneous axis if and only if the equations $A_3(u,v) = 0$ and $A_1(w) = A_2(w)$ hold. From these conditions we get the parametric representation

$$\hat{\mathbf{m}}(\tau) = \underbrace{\hat{\mathbf{p}}_0 - \frac{d}{2}(\vec{\mathbf{r}}\times\vec{\mathbf{q}})}_{(*)} + \frac{\varrho}{2\lambda}(\vec{\mathbf{q}}-\vec{\mathbf{r}}) + \tau(\vec{\mathbf{r}}+\vec{\mathbf{q}}) \qquad (\tau\in\mathbb{R}) \tag{32}$$

of the instantaneous axis at $s = 0$. The term $(*)$ represents the midpoint of the common perpendicular of the screw axes.

If $\varrho = 0$ holds, then the SGDS–motion (22) becomes a symmetric double–screw motion and the instantaneous axis passes through the common perpendicular of the screw axes. The latter property remains true even if the screw parameter $\mu, \nu$ of the screw motions are different.

The acceleration vector $\vec{\mathbf{a}}_p$ of the point $\hat{\mathbf{p}}$ is obtained from (23):

$$\begin{aligned}\vec{\mathbf{a}}_p = {} & \underbrace{\Big[(2\,\lambda\,\varrho - v\,\lambda^2)(\vec{\mathbf{r}},\vec{\mathbf{q}}) + \lambda^2\,u\,(1+2(\vec{\mathbf{r}},\vec{\mathbf{q}}))\Big]}_{=B_1(u,v)}\;\vec{\mathbf{r}} \\ + {} & \underbrace{\Big[-(2\,\lambda\,\varrho - v\,\lambda^2) - \lambda^2\,u\,(2+(\vec{\mathbf{r}},\vec{\mathbf{q}}))\Big]}_{=B_2(u,v)}\;\vec{\mathbf{q}} \\ + {} & \underbrace{\Big[2\,\lambda^2\,(1+(\vec{\mathbf{r}},\vec{\mathbf{q}}))\,w + \lambda^2\,d + 2\,\lambda\,\mu + 2\,\lambda^2\,d\,(\vec{\mathbf{r}},\vec{\mathbf{q}})\Big]}_{=B_3(w)}\;\vec{\mathbf{r}}\times\vec{\mathbf{q}}\,.\end{aligned} \tag{33}$$

The *acceleration pole* ($\vec{\mathbf{a}}_p = \vec{\mathbf{0}}$) at $s = 0$ can be found from $B_1(u,v) = B_2(u,v) = B_3(w) = 0$. It is given by (30) with

$$u = u_0 = 0\,, \quad v = v_0 = \frac{2\,\varrho}{\lambda} \text{ and } w = w_0 = \frac{d}{2}\cdot\frac{1+2(\vec{\mathbf{r}},\vec{\mathbf{q}})}{1+(\vec{\mathbf{r}},\vec{\mathbf{q}})} + \frac{\mu}{\lambda(1+(\vec{\mathbf{r}},\vec{\mathbf{q}}))}, \tag{34}$$

therefore it is contained in the plane spanned by the common perpendicular of the screw axes and by the second screw axis. Note that the acceleration pole depends on the parametrization of the motion! The equations (34) yield the acceleration pole with respect to the canonical parametrization introduced in (3).

In the case of a symmetric double screw motion ($\varrho = 0$) we have $v_0 = 0$, therefore the acceleration pole lies on the common perpendicular of the screw axes. As already observed by Hohenberg [5], this is also true in the case of different screw parameters $\mu, \nu$. This leads to a geometrical approach to Theorem 2: the acceleration pole of a double screw motion always lies on the common perpendicular of the screw axes, additionally the instantaneous axis passes through this line. Therefore the velocity vector at the acceleration pole is orthogonal to the common perpendicular of the screw axes (i.e., to the angular acceleration vector, cf. (8)). An arbitrary spatial motion does not fulfill this condition in general, therefore generally no osculating double screw motion can be found.

The inflection points (characterized by $\vec{\mathbf{v}}_p \| \vec{\mathbf{a}}_p$) of the SGDS–motion (22) at $s = 0$ form an algebraic curve of order three which can be obtained as the intersection of any two of the three quadric surfaces obtained from

$$\frac{A_1(w)}{A_2(w)} = \frac{B_1(u,v)}{B_2(u,v)}, \quad \frac{A_1(w)}{A_3(u,v)} = \frac{B_1(u,v)}{B_3(w)} \text{ and } \frac{A_2(w)}{A_3(u,v)} = \frac{B_2(u,v)}{B_3(w)}. \tag{35}$$

Consider the pencil of quadrics generated by any two of the three quadric surfaces (35). It contains the quadric surface

$$\frac{A_1(w)+A_2(w)}{A_3(u,v)} = \frac{B_1(u,v)+B_2(u,v)}{B_3(w)} \text{ resp. } \frac{a}{b_0 + b_1(u-v)} = \frac{c_0 + c_1(u-v)}{d_0 + d_1 w}, \tag{36}$$

with certain real coefficients $a, b_0, b_1, c_0, c_1, d_0, d_1 \in \mathbb{R}$ which can be obtained from straightforward calculations.

The quadric (36) is a parabolic cylinder whose generating lines are parallel to the instantaneous axis (i.e., to $\vec{\mathbf{r}} + \vec{\mathbf{q}}$). Moreover, its asymptotic direction is that of the common perpendicular of the screw axes ($\vec{\mathbf{r}} \times \vec{\mathbf{q}}$).

## Conclusion

As the main result, we have constructed a unique SGDS–motion which possesses a second order contact with $\mathrm{M}(t(s))$ at $s = 0$. This motion can said to be a kinematical analogue of the osculating circle. It may give a realistic impression of the local first and second order properties of a spatial motion.

Additionally it has been shown that generally no double screw motion exists which has this property. Therefore the composition of only two elementary spatial motions (i.e., of screw motions) does not yield the desired result. In contrast to this, in planar and spherical kinematics the composition of two elementary motions leads to a analogue of the osculating circle. In those cases the constructions immediately result from the Euler–Savary formula.

Using methods from Lie theory, a recent article by Wallner [14] discusses the approximation of motions by composing elementary motions in several classical motion groups, including those of planar and spatial Euclidean kinematics. He derived some criteria on the group which guarantee the existence of osculating motions composed of two or three elementary motions.

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## Eine Schmiegbewegung zweiter Ordnung für räumliche euklidische Bewegungsvorgänge

**Zusammenfassung.** In der Arbeit wird versucht, ein kinematisches Analogon zum Schmiegkreis einer Kurve zu konstruieren. Als Hauptergebnis gelingt es dabei, eine eindeutig bestimmte *symmetrische gleitende Doppelschraubung* zu finden, die eine gegebene räumliche Bewegung an einer Stelle von zweiter Ordnung berührt. Diese gleitende Doppelschraubung vermittelt eine anschauliche Deutung der lokalen Eigenschaften erster und zweiter Ordnung des gegebenen Bewegungsvorgangs. Zusätzlich wird gezeigt, daß im allgemeinen keine Doppelschraubung mit einer ähnlichen Eigenschaft existiert.