Approximate Parameterization of Planar Cubics

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Abstract. This paper considers the problem of approximating a given segment of an algebraic cubic curve by a cubic rational curve satisfying G^1 boundary conditions. According to the classical non-linear approximation theory of Meinardus and Schwedt [2,6], the existence and uniqueness of solutions is guaranteed if the deviation functions satisfy both the local and the global Haar property. We show that the local Haar property is satisfied and give a counterexample for the global one. In addition, we describe a Remes-type algorithm for computing a solution.

§1. Introduction

In order to switch from the implicit form of a planar algebraic curve to a rational parametric representation, one has to solve the parameterization problem. According to the classical results from algebraic geometry, exact rational parameterizations are available only for algebraic curves of genus zero. Consequently, the existing symbolic techniques for parameterization are useful only for a relatively small subclass of these curves. Besides, they will almost always fail if the coefficients of the algebraic curves are given in floating point form. In applications in geometric design, however, one is often interested only in a small segment of the algebraic curve, and not in the curve as a whole. Therefore, an algorithm for approximate parameterization of algebraic curve segments is needed.

This paper discusses the rational cubic case with G^1 boundary conditions. We consider an algebraic cubic curve segment in Bernstein–Bézier representation [7]

$$f(u,v,w) = \sum_{\substack{i,j,k \in \mathbb{Z}_+ \\ i+j+k=3}} B_{i,j,k}(u,v,w)\, b_{i,j,k}, \tag{1}$$

Curve and Surface Fitting: Saint-Malo 2002
XXX (eds.), pp. 1–10.

ISBN XXX.

with certain coefficients $b_{i,j,k} \in \mathbb{R}$, where $B_{i,j,k}$ are the bivariate Bernstein polynomials and u, v, w the barycentric coordinates with respect to an associated domain triangle $\triangle$ with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^2$. Throughout this paper we assume that the algebraic cubic curve segment

$$\mathcal{F} := \{(x, y) \in \triangle\mathbf{ABC} \mid f(x, y) = 0\}$$

is a regular curve segment with end points $\mathbf{A}$ and $\mathbf{C}$. In addition, the edges $\mathbf{AB}$ and $\mathbf{BC}$ are assumed to be tangent to the curve segment. As a necessary (but not sufficient) condition, the coefficients satisfy

$$b_{0,0,3} = b_{0,1,2} = b_{2,1,0} = b_{3,0,0} = 0. \tag{2}$$

Under these assumptions, there exists a (not necessarily rational) C^2 parameterization $\mathbf{x} : [a, b] \to \mathbb{R}^2$ of $\mathcal{F}$. We denote with $\mathbf{n}(t)$ the normal vector and with $\kappa(t)$ the curvature at $\mathbf{x}(t)$. Clearly, this representation is generally non–rational. We are interested in generating an approximate cubic *rational* parametric representation, which also preserves boundary points and boundary tangents.

In this paper we formulate this task as a rational approximation problem; the approximate parameterization is to be found by minimizing the distance between the original curve and the parametric one. For the convenience of the reader, we summarize some basic notions which were introduced by Degen [2]:

A curve $\mathcal{G}$ which is given by a C^1 parametric representation $\mathbf{y} : [c, d] \to \mathbb{R}^2$, is called admissible with respect to $\mathbf{x} : [a, b] \to \mathbb{R}^2$, if the following conditions are satisfied:

(*i*) For each $s \in [c, d]$ exist real values $t \in [a, b]$ and ρ, such that

$$\mathbf{y}(s) = \mathbf{x}(t) + \rho\,\mathbf{n}(t) \quad \text{where} \quad \rho\,\kappa(t) < 1 \tag{3}$$

(*ii*) For each $t \in [a, b]$ there is exactly one $s = \sigma(t) \in [c, d]$ satisfying (3).

(*iii*) The tangent vector $\dot{\mathbf{y}}(\sigma(t))$ is linearly independent of $\mathbf{n}(t)$.

(*iv*) The segment end points of both curves $\mathcal{F}$ and $\mathcal{G}$ are identical, i.e. $\mathbf{x}(a) = \mathbf{y}(c), \mathbf{x}(b) = \mathbf{y}(d)$.

Each admissible curve has an associated reparameterization

$$\sigma : [a, b] \to [c, d], \quad t \mapsto \sigma(t)$$

(cf. (*i*)), and an associated deviation function $\rho : [a, b] \to \mathbb{R}$ such that

$$\mathbf{y} \circ \sigma = \mathbf{x} + \rho\,\mathbf{n} \quad \text{or, equivalently,} \quad \mathbf{y}(\sigma(t)) = \mathbf{x}(t) + \rho(t)\,\mathbf{n}(t). \tag{4}$$

Finally, the maximum absolute value of the deviation function

$$d_N(\mathcal{F}, \mathcal{G}) := \max_{t \in [a,b]} |\rho(t)|$$

is called the normal distance of $\mathcal{G}$ from $\mathcal{F}$.

Using these notations we formulate the following rational approximation problem (RAP), see Fig. 1:

Problem 1. *Given a domain triangle $\triangle\mathbf{ABC}$ and a a single arc of a cubic curve satisfying G^1 boundary conditions, find an admissible cubic rational Bézier curve*

$$\mathbf{y} : [0,1] \to \mathbb{R}^2, \;\; s \mapsto \mathbf{y}(s)$$

interpolating both the boundary points and the associated tangents, which minimizes the normal distance of $\mathcal{G}$ from $\mathcal{F}$.

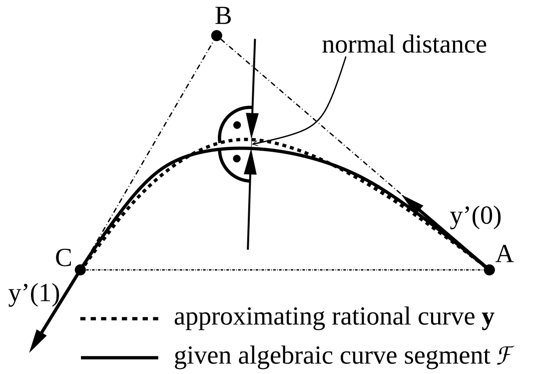


Fig. 1. Rational approximation with G^1 boundary conditions.

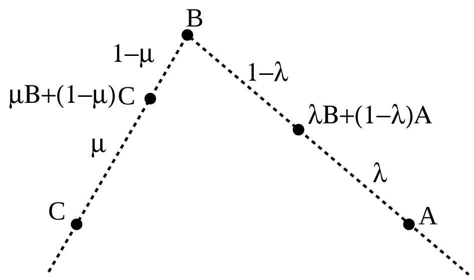


Fig. 2. Geometrical meaning of λ and μ.

Using the the nonlinear approximation theory of Meinardus and Schwedt [6], the analogous problem for polynomial curves of arbitrary degree has thoroughly been discussed by Degen [2] and Eisele [3]. As observed there, polynomial Bézier curves satisfy the so–called local Haar property. The global Haar property is satisfied for quadratic polynomial Bézier curves and for cubic polynomial Bézier curves with G^1 boundary conditions. Moreover, it has been shown that the global Haar property for polynomial Bézier curves is satisfied for arbitrary degree and G^k boundary conditions, but only in some vicinity of a given curve [3]. Consequently, in the polynomial case, the approximation problem has a locally unique solution, which can then be computed by a Remes–type algorithm (see [2, 8]).

§2. Existence and uniqueness of a best approximation

The cubic rational curves which satisfy the G^1 Hermite boundary condition have the Bézier representation

$$\mathbf{y}(s) = \left(\frac{X(s)}{W(s)}, \frac{Y(s)}{W(s)}\right)^\top \tag{5}$$

with

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = (1-s)^3\,\mathbf{A} + 3\,w_1\,(1-s)^2\,s\,(\lambda\,\mathbf{B} + (1-\lambda)\,\mathbf{A}) + 3\,w_2\,(1-s)\,s^2\,(\mu\,\mathbf{B} + (1-\mu)\,\mathbf{C}) + s^3\mathbf{C} \tag{6}$$

and

$$W(s) = (1-s)^3 + 3\,w_1\,(1-s)^2\,s + 3\,w_2\,(1-s)\,s^2 + s^3. \qquad (7)$$

Here, we use the so–called standard representation ($w_0 = w_3 = 1$), which can be obtained using a suitable bilinear parameter transformation, see [4]. Hence for given boundary values, the set $\mathcal{R}$ of cubic rational Bézier curves satisfying the G^1 boundary conditions depends on the four parameters λ, μ, w_1 and w_2. This defines a bijective mapping

$$\Lambda : \mathbb{R}^4 \to \mathcal{R} \quad (\lambda, \mu, w_1, w_2) \mapsto \mathbf{y}(.). \qquad (8)$$

Let m be the number of free parameters (in our case, $m = 4$). The mapping (8) induces a differentiable structure on the set $\mathcal{R}$. Now, taking the admissibility conditions into account, we denote – for a given curve $\mathcal{F}$ – the set of all admissible curves by $\mathcal{R}'$. It is the image of an open subset $M \subset \mathbb{R}^4$ under the mapping Λ. A proof for this fact is given in [2] for the polynomial case. Since the proof does not use any special features of polynomial curves, it can be adapted to the rational situation.

If we consider only the admissible curves in $\mathcal{R}$, then both the differentiable structure and the dimension m are maintained. The differential structure can be described by the mapping

$$\Lambda' := \Lambda|_M.$$

Now we consider the reparameterization and deviation functions σ, ρ in (4) not only for a single curve $\mathcal{G}$, but for all admissible curves in $\mathcal{R}'$. Using the implicit function theorem we obtain differentiable functions

$$\hat{\rho}, \hat{\sigma} : M \times [a,b] \to \mathbb{R},$$

such that the specializations

$$t \mapsto \rho_{\mathbf{p}}(t) = \hat{\rho}(\mathbf{p}, t) \;\text{ resp. }\; t \mapsto \sigma_{\mathbf{p}}(t) = \hat{\sigma}(\mathbf{p}, t)$$

to a specific $\mathbf{p} = (\lambda, \mu, w_1, w_2) \in M$ are the deviation and reparameterization functions of the rational Bézier curve $\Lambda'(\mathbf{p}) \in \mathcal{R}'$.

We also define a map Γ which assigns the deviation function of the corresponding admissible curve to each point in the parameter domain,

$$\Gamma : M \to C[0,1], \quad \mathbf{p} \mapsto \rho_{\mathbf{p}}$$

According to [2], the set $\mathcal{M} = \Gamma(M)$ is a differentiable manifold.

The theory of Meinardus and Schwedt (see [1,6]) is concerned exactly with this situation: given $\hat{\rho} : M \times [a,b] \to \mathbb{R}$ as above and $f \in C[a,b]$, find $\mathbf{p}_0 \in M$ such that

$$\|\hat{\rho}(\mathbf{p}_0, .) - f\| = \inf_{\mathbf{p} \in M} \|\hat{\rho}(\mathbf{p}, .) - f\|.$$

The function $\hat{\rho}(\mathbf{p}_0, .)$ is called best approximation to f with respect to the parameter domain M. In our case, $f \equiv 0$. For the convenience of the reader, we summarize some notions and results from this theory (see [2] for more details):

(1) The function $\hat{\rho}(\mathbf{p}_0, .)$ of $\mathcal{M}$ is called local best approximation to f, if there exists an open subset $U \subset M$ with $\mathbf{p}_0 \in U$ such that $\hat{\rho}(\mathbf{p}_0, .)$ is a best approximation to f with respect to the parameter domain $M \cap U$.

(2) The elements of the linear tangent space $T_{\mathbf{p}}\mathcal{M}$ of $\mathcal{M}$ at $\Gamma(\mathbf{p})$ can be represented by linear combinations of the m partial derivatives of Γ (in our case with respect to λ, μ, w_1, w_2), evaluated at $\mathbf{p}$:

$$T_{\mathbf{p}}\mathcal{M} = \operatorname{span}\left\{\frac{\partial\Gamma}{\partial\lambda}(\mathbf{p}), \frac{\partial\Gamma}{\partial\mu}(\mathbf{p}), \frac{\partial\Gamma}{\partial w_1}(\mathbf{p}), \frac{\partial\Gamma}{\partial w_2}(\mathbf{p})\right\}$$

(3) $\mathcal{M}$ is said to satisfy the local Haar property at $\mathbf{p}$, if the linear tangent space $T_{\mathbf{p}}\mathcal{M}$ satisfies the classical Haar property, i.e. if each function $\tau \in T_{\mathbf{p}}\mathcal{M}$, $\tau \neq 0$, has at most $m-1$ zeros.

(4) $\mathcal{M}$ is said to satisfy the global Haar property, if $\rho_{\mathbf{p}} - \rho_{\mathbf{q}}$ has at most $m-1$ zeros for every pair $\mathbf{p}, \mathbf{q} \in M$.

(5) A continuous function $r : [a, b] \to \mathbb{R}$ is said to be an alternant with $m+1$ extremal points, iff there are $m+1$ points

$$a \leq x_1 < \ldots < x_{m+1} \leq b,$$

such that $|r(x_i)| = \|r\|_\infty$ for $i = 1, \ldots, m+1$ and $r(x_i) = -r(x_{i+1})$ for $i = 1, \ldots, m$.

Using these notions, the following results can be formulated.

Theorem 2 (Meinardus and Schwedt, [6]). *Let $f \in C[a, b]$ and $\mathcal{M}$ be induced by a global differentiable function $\hat{\rho} : M \times [a, b] \to \mathbb{R}$ (as above) and satisfy the global Haar property. If there is a $\mathbf{p} \in M$ such that the local Haar property is satisfied at $\mathbf{p}$ and $\rho_{\mathbf{p}} - f$ is an alternant with $m+1$ extremal points, then $\rho_{\mathbf{p}}$ is the unique best approximation to f.*

If $\mathcal{M}$ does not satisfy the global Haar property, we can formulate a necessary condition for a local best approximation (see [1]):

Remark 3. *Let $f \in C[a, b]$ and $\mathcal{M}$ be induced by a global differentiable function $\hat{\rho} : M \times [a, b] \to \mathbb{R}$. If $\hat{\rho}(\mathbf{p}_0, .)$ is a local best approximation to f, then $\rho_{\mathbf{p}}$ is an alternant with $m+1$ extremal points.*

Note that in our case $m = 4$. From the G^1 boundary conditions, every $\rho_{\mathbf{p}}$ has two–fold zeros at 0 and 1. These zeros must not be counted, neither in the local Haar property nor in the global one (see [2], Remark after Definition 4).

§3. Local Haar condition

Now we are ready to apply this theory to our special situation.

Theorem 4. *The differential manifold $\mathcal{M}$ above satisfies the local Haar property.*

Proof: Consider an arbitrary point $\mathbf{p} = (\lambda, \mu, w_1, w_2) \in M \subset \mathbb{R}^4$, along with the associated rational Bézier curve $\mathbf{y} = \Lambda'(\mathbf{p}) \in \mathcal{R}$. Let $\tau \in T_{\mathbf{p}}\mathcal{M}$ be an arbitrary function from the linear tangent space of $\mathcal{M}$ at a point $\hat\rho(\mathbf{p}, .) \in \mathcal{M}$. We have to show that $\tau = \tau(t)$ has at most 3 zeros in $]0, 1[$. This function can be written as

$$\tau(t) = (D_{\mathbf{v}}\hat\rho)(\mathbf{p}, t),$$

for some direction $\mathbf{v}$ in the parameter space $\mathbb{R}^4$, where $D_{\mathbf{v}}$ is the directional derivative. In order to estimate the number of zeros of $\tau(t)$ we use a technique due to Degen (see [2]). Since $\mathbf{y}$ depends on $\mathbf{p}$, $\mathbf{y}_{\mathbf{p}}(t) = \mathbf{y}(\mathbf{p}, t)$, we may rewrite (4) as

$$\mathbf{y}_{\mathbf{p}}(\sigma_{\mathbf{p}}(t)) = \mathbf{x}(t) + \underbrace{\rho_{\mathbf{p}}(t)}_{=\hat\rho(\mathbf{p},t)} \; \mathbf{n}(t) \tag{9}$$

Applying the directional derivative $D_{\mathbf{v}}$ and using the chain rule yields

$$(D_{\mathbf{v}}\mathbf{y})(\mathbf{p}, \sigma_{\mathbf{p}}(t)) + \mathbf{y}'_{\mathbf{p}}(\sigma_{\mathbf{p}}(t)) \; (D_{\mathbf{v}}\hat\sigma)(\mathbf{p}, t) = \underbrace{(D_{\mathbf{v}}\hat\rho)(\mathbf{p}, t)}_{=\tau(t)} \; \mathbf{n}(t). \tag{10}$$

Using the abbreviation $[\mathbf{u}, \mathbf{v}] = u_1 v_2 - u_2 v_1$ for the determinant of two vectors, we get by multiplying both sides with the tangent vector $\mathbf{y}'_{\mathbf{p}}$

$$[(D_{\mathbf{v}}\mathbf{y})(\mathbf{p}, \sigma_{\mathbf{p}}(t)), \; \mathbf{y}'_{\mathbf{p}}(\sigma_{\mathbf{p}}(t))] = \tau(t) \; \underbrace{[\mathbf{n}(t), \; \mathbf{y}'_{\mathbf{p}}(\sigma_{\mathbf{p}}(t))]}_{(*)}. \tag{11}$$

Due to the admissibility assumption (iii), the determinant $(*)$ has no zeros in $]a, b[$. Consequently, the zeros of τ are the same as those of the left–hand side in (11). The parameter transformation $t \mapsto s(t) = \hat\sigma(\mathbf{p}, t)$ can be omitted, since it is a diffeomorphism. Consequently, the numbers of zeros of $\tau(t)$ in $]a, b[$ and of

$$T(s) := [(D_{\mathbf{v}}\mathbf{y})(\mathbf{p}, s), \mathbf{y}'_{\mathbf{p}}(s)] \quad \text{in }]0, 1[$$

are equal.

Computationally, $T(s)$ is a rational expression in the variables $s, \lambda, \mu, w_1, w_2$, in the coordinates v_1, v_2, v_3, v_4 of the direction $\mathbf{v}$, and in $a_1, a_2, b_1,$

b_2, c_1, c_2 (the coordinates of $\mathbf{A}, \mathbf{B}, \mathbf{C}$). We calculated this expression explicitly with Maple and factorized it. It turned out that

$$T(s) = \frac{18\, c\, s^2 (1-s)^2\, q(s)}{W(s)^3},$$

where c is the oriented area of the triangle $\Delta\mathbf{ABC}$ and

$$\begin{aligned} q(s) = {} & (\lambda\ (1-\mu)\,(w_1 v_4 - 2\,w_2 v_3) - w_1 w_2\,(2\ (1-\mu)\,v_1 + v_2\lambda))\,(1-s)^3 \\ & + \left(w_1\,(-3\,w_1 w_2\,(1-\mu) - 1)\,v_1 - v_3\lambda - 2\,{w_2}^2 v_2\right)(1-s)^2\,s \\ & + \left(-2\,v_1 {w_1}^2 - w_2\,(3\,w_1 w_2\,(1-\lambda) + 1)\,v_2 - \mu\,v_4\right)(1-s)\,s^2 \\ & + (\mu\ (1-\lambda)\,(w_2 v_3 - 2\,w_1 v_4) - w_1 w_2\,(v_1 \mu + 2\ (1-\lambda)\,v_2))\,s^3 \end{aligned}$$

is a cubic polynomial in s. Since the points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are assumed to be non–collinear, we have $c \neq 0$ and therefore $T(s)$ and $\tau(t)$ have at most 3 zeros in $]0,1[$ resp. in $]a,b[$. □

§4. Global Haar condition

If $\mathcal{M}$ satisfied the global Haar property (for some given curve $\mathcal{F}$), there would not exist any two deviation functions $\rho_\mathbf{p}$ and $\rho_\mathbf{q}$ such that $\rho_\mathbf{p} - \rho_\mathbf{q}$ has 4 zeros. Clearly, these zeros correspond to the intersection points of $\Lambda'(\mathbf{p})$ and $\Lambda'(\mathbf{q})$. Therefore, any two admissible curves $\Lambda'(\mathbf{p})$ and $\Lambda'(\mathbf{q})$ would not intersect in more than three points.

Unfortunately, as demonstrated by the following example, this assumption is not satisfied by rational Bézier cubics.

Example. Let the $\mathbf{A} := (1,0), \mathbf{B} := (0,1), \mathbf{C} := (0,0)$ and

$$\begin{aligned} \mathbf{p} = (\lambda_1, \mu_1, w_{1,1}, w_{1,2}) &:= (\tfrac{52881}{100000}, \tfrac{18807}{25000}, \tfrac{20373}{25000}, \tfrac{82381}{100000}) \\ \mathbf{q} = (\lambda_2, \mu_2, w_{2,1}, w_{2,2}) &:= (\tfrac{11233}{20000}, \tfrac{33023}{50000}, \tfrac{98521}{100000}, \tfrac{24947}{25000}). \end{aligned}$$

Using (5)–(7) we obtain two cubic rational Bézier curves $\mathbf{y_p}(s) = \Lambda'(\mathbf{p})$ and $\mathbf{y_q}(t) = \Lambda'(\mathbf{q})$, which satisfy G^1 boundary conditions. Both curves are also admissible with respect to a suitable algebraic cubic (for instance, this curve can be obtained by implicitizing $\mathbf{y_p}$). These curve segments have – in addition to the end points – the four intersections

$$(0.667, 0.290),\ (0.412, 0.431),\ (0.192, 0.445),\ (0.038, 0.273).$$

This can also be seen by inspecting the deviation function of $\mathbf{y_q}$ with respect to $\mathbf{y_p}$, see Fig. 3. In addition to this function, the figure shows the curves $\mathbf{y_q}$ and $\mathbf{y_q} + c\,\rho(t)\,\mathbf{n}(t)$ (i.e., second curve with exaggerated distance).

According to a result of Eisele [3], the global Haar condition is locally satisfied in the polynomial case, within a certain neighbourhood of a given curve. So far, we did not succeed in generalizing this result to the rational situation.

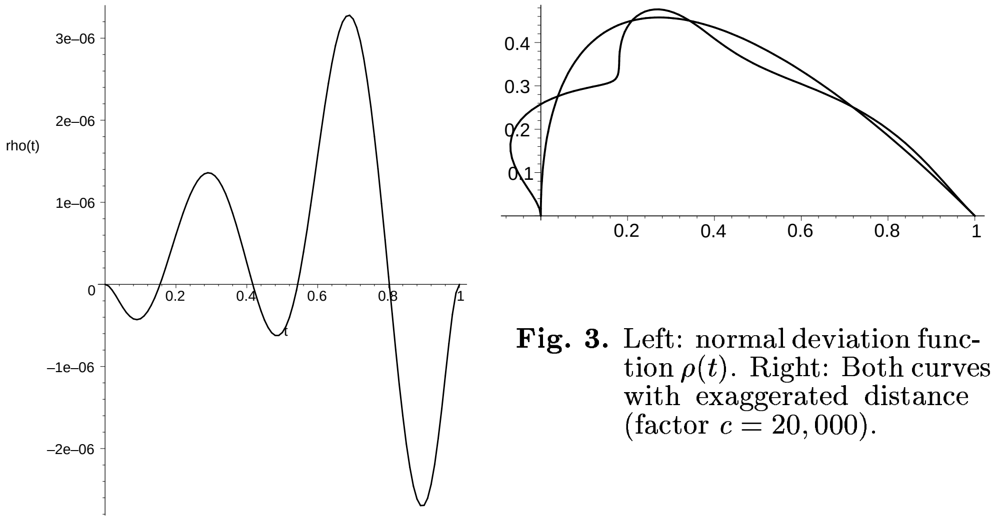


Fig. 3. Left: normal deviation function $\rho(t)$. Right: Both curves with exaggerated distance (factor $c = 20,000$).

§5. Remes-type algorithm

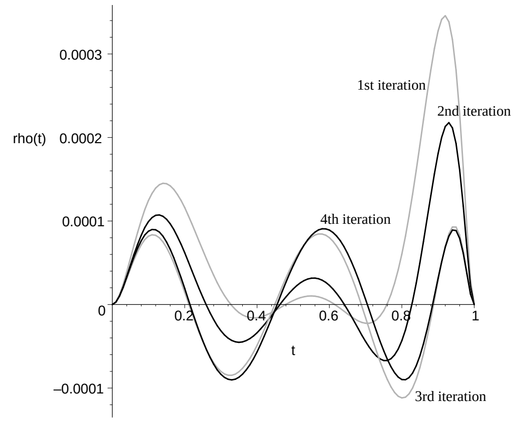
Unfortunately, the theorem of Meinardus and Schwedt is not applicable to our problem (RAP), since $\mathcal{M}$ does not satisfy the global Haar property. Nevertheless, for practical purposes it is useful to compute an alternating approximation, but one cannot guarantee that it is the best solution of the RAP. The ideas of Remes' algorithm can be generalized in order to construct an alternating approximating curve. More details will be given in the forthcoming paper [5]. Here we will give an outline of our approach.

Note that it is not convenient to use the rational Bézier representation for the approximating curves $\mathbf{y}(s)$. Using this parametric representation, one would have to solve systems of polynomial equations, and the choice of a suitable initial solution may cause serious problems. We bypass these problems by resorting to the the so–called Bézier coefficient space. Every cubic algebraic curve can be identified by its bivariate Bézier coefficients (see equation (1)). Clearly, this is a homogeneous representation, since multiplying all bivariate Bézier coefficients with the constant does not change the curve. Thus, in the case of G^1 boundary conditions, the curve can be identified with the point

$$(B_{0,2} : \ B_{0,3} : \ B_{1,0} : \ B_{1,1} : \ B_{1,2} : \ B_{2,0}) \in \mathbb{P}^5$$

from a 5–dimensional real projective space.

Next we derive an algebraic condition for the rationality of a cubic curve. In the case of G^1 boundary conditions, it is a homogeneous polynomial of degree 8 in the Bézier coefficients, which is called the discriminant. This polynomial defines a hypersurface $\mathcal{D} \subset \mathbb{P}^5$, which contains all points which correspond to (irreducible) rational cubics. Using this geometric approach it is now relatively simple to construct rational cubic curves interpolating certain points, which is a basic step in the Remes–type algorithm. In order to keep the equations simple, it is convenient to compute



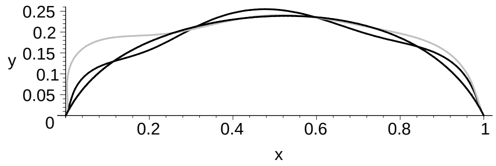


Fig. 4. Left: Normal deviation functions of the solutions with respect to $\mathcal{F}$; Right: Given curve segment $\mathcal{F}$, along with initial and final solution, with exaggerated distance (factor c=200).

the iteration points in the Bézier coefficient space and then to project the results onto the discriminant $\mathcal{D}$, in order to obtain rational curves. By modifying the points on the algebraic curve, we are able to achieve the desired alternating behaviour of the deviation function. This is demonstrated by the following example.

Example. Consider the domain triangle $\mathbf{A} = (1, 0)$, $\mathbf{B} = (\frac{1}{2}, 1)$, $\mathbf{C} = (0, 0)$ and the and the algebraic curve

$$f(x, y) = \frac{5}{8} B_{0,2,1} + B_{0,3,0} - \frac{5}{16} B_{1,0,2} + \frac{1}{8} B_{1,1,1} + B_{1,2,0} - \frac{9}{16} B_{2,0,1},$$

where $B_{i,j,k} = B_{i,j,k}(u, v, w)$ are the cubic bivariate Bernstein polynomials with respect to the domain triangle.

Using the Remes–type algorithm we generate a 4 times alternating cubic rational Bézier curve with respect to this given curve segment $\mathcal{F}$. Fig. 4 (left) shows the normal deviations of $g_1, \ldots g_4$ with respect to the given curve segment $\mathcal{F}$. Of course, the extremal values of the deviation functions correspond to the normal distances. They decrease from $.3458\,10^{-3}$ to $.9019\,10^{-4}$. The alternating behaviour can clearly be seen.

§6. Concluding remarks

In this paper we have used the nonlinear approximation theory of Meinardus and Schwedt to analyze the problem of generating an approximate rational parametric representation of a given algebraic cubic curve. As demonstrated in this paper, the situation is different from the polynomial case, as the global Haar property is not satisfied by rational cubics with G^1 boundary conditions. Thus, the assumptions which would guarantee the uniqueness of the solutions are not satisfied.

We developed a Remes-type algorithm which produces solutions which satisfy the necessary condition for optimality. This algorithm, which is based on the implicit representation and rationality criteria, will be described in more detail in a forthcoming paper [5]. The generalization of our approach to the surface case is currently under consideration.

Acknowledgments. This research has been supported by Austrian science fund (FWF) in the frame of the Special Research Programme (SFB) F013 "Numerical and Symbolic Scientific Computing", project 15.

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