Linear convexity conditions for parametric tensor–product Bézier surface patches

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Abstract. In the first part of this article we present a method for generating linear sufficient conditions which guarantee the convexity of parametric tensor–product Bézier surfaces. The linearized conditions can be adapted to any strongly convex surface patch.
In the second part we discuss the so–called lifting problem for convex surfaces. With the help of the linearized convexity conditions this task can be formulated as a simple optimization problem.

1 Introduction

Tensor–product Bézier and B-spline surfaces are widely used for the description of free–form surfaces in CAD systems. Convexity of surfaces is an important feature for many applications, e.g., in the automotive or in the ship–building industry. In order to derive algorithms for the construction (e.g., by interpolation or approximation of given data) or for the modification (e.g., "lifting" of these surfaces) of convex surfaces, appropriate constraints ensuring the convexity of the patches have to be found.

Very strong sufficient convexity conditions for parametric tensor–product Bézier surfaces have been derived in Schelske's Ph.D. thesis [11] (see also [7], p. 263). These conditions are fulfilled only by convex translational surfaces (i.e., by convex surfaces which are swept out by the motion of a rigid contour curve). Convexity criteria for parametric triangular Bézier surfaces were developed by Zhou [16]. These conditions lead to systems of inequalities of degree 3 and 6 in the components of the control points.

More research has been done in the case of (piecewise) polynomial functions. A survey of convexity criteria for such functions can be found in the articles by Dahmen [1] and Goodman [6]. As an application of such conditions, approximation by convex piecewise quadratic polynomials on Powell–Sabin splits is discussed in [15]. Most of these criteria apply to polynomials in Bernstein–Bézier representation over triangles. A recent article by Floater [5] discusses convexity conditions for functional tensor–product Bézier– and B-spline surfaces. It is shown how the convexity of such functions can be guaranteed with the help of a set of *quadratic* inequalities for the Bézier coefficients.

In the remainder of this article we derive a method for constructing *linear* sufficient convexity conditions for parametric tensor–product Bézier surfaces. The control polygons of the first and second derivatives are bounded by appropriate wedges. These wedges yield a set of linear inequalities for the components of the control points. The bounding wedges for the derivatives are chosen with the help of a convex reference surface.

Using the linear constraints obtained from the method, several tasks of convexity preserving surface construction or modification can be formulated as optimization problems with linear constraints. As an example we discuss the problem of the convexity–preserving "lifting" of a single Bézier surface patch. This problem has been discussed by Schichtel in his Ph.D. thesis [12], see also [13]. We present a new approach which is based on the linearized convexity conditions.

2 Preliminaries

Consider a tensor–product Bézier surface patch of degree (m, n),

$$\mathbf{x}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u)\ B_j^n(v)\ \mathbf{b}_{i,j} \quad (0 \le u, v \le 1) \tag{2.1}$$

with the control points $\mathbf{b}_{i,j} \in \mathbb{R}^3$ and the well–known Bernstein polynomials $B_s^r(t) = \binom{r}{s} t^s (1-t)^{r-s}$, see [7]. The first and second partial derivative vectors of this surface are

$$\mathbf{x}_\diamond(u,v) = \sum_{i=0}^{m_\diamond} \sum_{j=0}^{n_\diamond} B_i^{m_\diamond}(u)\ B_j^{n_\diamond}(v)\ \Delta_\diamond \mathbf{b}_{i,j}, \quad \diamond \in \{u, v, uu, uv, vv\} \tag{2.2}$$

with

$$\begin{array}{lll} \Delta_u \mathbf{b}_{i,j} = m\ (\mathbf{b}_{i+1,j} - \mathbf{b}_{i,j}), & m_u = m{-}1, & n_u = n, \\ \Delta_v \mathbf{b}_{i,j} = n\ (\mathbf{b}_{i,j+1} - \mathbf{b}_{i,j}), & m_v = m, & n_v = n{-}1, \\ \Delta_{uu} \mathbf{b}_{i,j} = (m{-}1)\ (\Delta_u \mathbf{b}_{i+1,j} - \Delta_u \mathbf{b}_{i,j}), & m_{uu} = m{-}2, & n_{uu} = n, \\ \Delta_{uv} \mathbf{b}_{i,j} = m\ (\Delta_v \mathbf{b}_{i+1,j} - \Delta_v \mathbf{b}_{i,j}), & m_{uv} = m{-}1, & n_{uv} = n{-}1 \quad \text{and} \\ \Delta_{vv} \mathbf{b}_{i,j} = (n{-}1)\ (\Delta_v \mathbf{b}_{i,j+1} - \Delta_v \mathbf{b}_{i,j}), & m_{vv} = m, & n_{uu} = n{-}2. \end{array} \tag{2.3}$$

The curvature properties of this surface at a point $(u, v) \in [0, 1]^2$ are governed by its first and second fundamental forms, see [7] or any textbook on differential geometry. The coefficients $(g_{r,s})_{r,s=1,2} = (g_{r,s}(u, v))_{r,s=1,2}$ of the first fundamental form are

$$g_{1,1} = \mathbf{x}_u \circ \mathbf{x}_v,\ \ g_{1,2} = g_{2,1} = \mathbf{x}_u \circ \mathbf{x}_v,\ \ g_{2,2} = \mathbf{x}_v \circ \mathbf{x}_v, \tag{2.4}$$

whereas the coefficients $(L_{r,s})_{r,s=1,2} = (L_{r,s}(u, v))_{r,s=1,2}$ of the second fundamental form result from

$$L_{1,1} {=} \frac{1}{D}\ [\mathbf{x}_{uu}, \mathbf{x}_u, \mathbf{x}_v],\ \ L_{1,2} {=} L_{2,1} {=} \frac{1}{D}\ [\mathbf{x}_{uv}, \mathbf{x}_u, \mathbf{x}_v],\ \ L_{2,2} {=} \frac{1}{D}\ [\mathbf{x}_{vv}, \mathbf{x}_u, \mathbf{x}_v], \tag{2.5}$$

with $D = D(u, v) = \|\mathbf{x}_u \times \mathbf{x}_v\|$. (We use the notations $\mathbf{x} \circ \mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$ for the inner and cross products of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, respectively. In addition, $[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \mathbf{x} \circ (\mathbf{y} \times \mathbf{z})$ and $\|\mathbf{x}\| = \sqrt{\mathbf{x} \circ \mathbf{x}}$.)

At a surface point $\mathbf{x}(u, v)$ $(0 \le u, v \le 1)$ we may consider the tangent

$$\mathbf{x}(u,v) + \tau\ \left(\ \xi\ \mathbf{x}_u(u,v) + \eta\ \mathbf{x}_v(u,v)\ \right) \quad (\tau \in \mathbb{R}). \tag{2.6}$$

Its direction is specified by the real coefficients ξ, η. Associated with the tangent we have the normal curvature (i.e., the curvature of the normal section through the surface with this direction)

$$\kappa_n(u,v,\xi,\eta) = \frac{\xi^2\ L_{1,1}(u,v) + 2\ \xi\ \eta\ L_{1,2}(.) + \eta^2\ L_{2,2}(.)}{\xi^2\ g_{1,1}(u,v) + 2\ \xi\ \eta\ g_{1,2}(.) + \eta^2\ g_{2,2}(.)} \quad (\xi, \eta) \ne (0,0) \tag{2.7}$$

We are interested in methods for constructing or modifying surface patches (2.1) with always non–negative normal curvature at all points. (Equivalently we may consider surfaces with always non–positive normal curvature. Swapping the parameters u, v changes the sign of the normal curvature.) Such surfaces have non–negative Gaussian curvature and are locally convex, i.e., they are convex in an appropriate neighbourhood of any one of its inner points (provided that they are regular ($D \ne 0$) at this point).

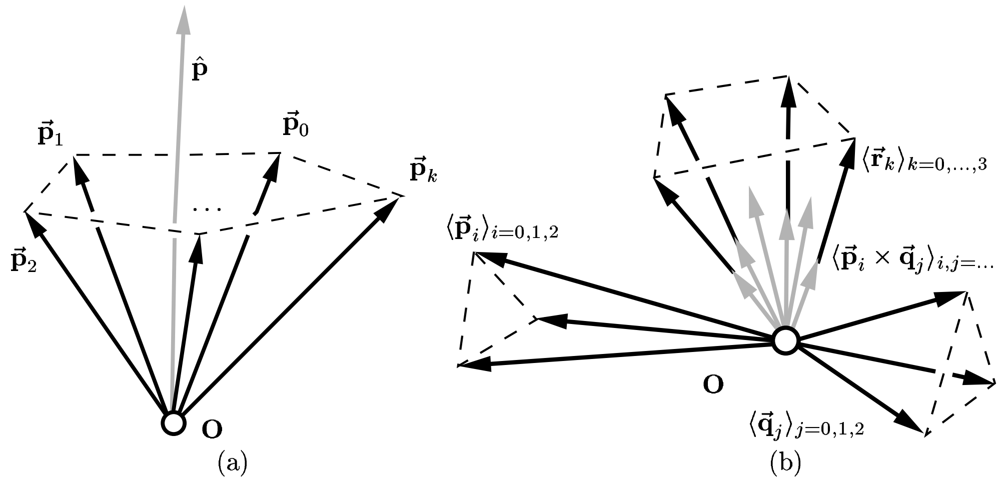

FIGURE 1. A wedge and its bounding vectors (a) and the cross product of two wedges (b).

The first fundamental form is well known to be positive definite. Therefore, at a regular surface point ($D(u,v) \neq 0$) having only non–negative normal curvature $\kappa_n(u,v,\xi,\eta) \geq 0$ is equivalent to

$$\left[\xi^2\,\mathbf{x}_{uu}(u,v)+2\,\xi\,\eta\,\mathbf{x}_{uv}(.)+\eta^2\,\mathbf{x}_{vv}(.)\ ,\mathbf{x}_u(.),\mathbf{x}_v(.)\right] \geq 0. \tag{2.8}$$

for all $\xi,\eta \in \mathbb{R}$. We will present conditions for the control points $\mathbf{b}_{i,j}$ of the surface (2.1) which guarantee this property for all points. The expression (2.8) is a quadratic form in ξ and η.

In order to describe bounds for the first and second partial derivatives we will use the following notions. Consider a sequence of unit vectors $(\vec{\mathbf{p}}_i)_{i=0,\dots,k}$ from $S^2 \subset \mathbb{R}^3$ $(k \geq 2)$. This sequence will be called a *(ordered) set of unit bounding vectors of a wedge*, denoted by

$$\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, \cdots, \vec{\mathbf{p}}_k \rangle, \tag{2.9}$$

if it satisfies the following three conditions (cf. Figure 1a):

(a) There exists a vector $\hat{\mathbf{p}} \in \mathbb{R}^3$ such that the inequalities $\vec{\mathbf{p}}_i \circ \hat{\mathbf{p}} > 0$ hold for $i = 0,\dots,k$. Thus, all vectors $\vec{\mathbf{p}}_i$ lie on one side of the plane perpendicular to $\hat{\mathbf{p}}$ which passes through the origin $\mathbf{O}$.

(b) The sequence contains no triple of linearly dependent vectors.

(c) The inequality $[\vec{\mathbf{p}}_j, \vec{\mathbf{p}}_i, \vec{\mathbf{p}}_{(i+1)\mathrm{mod}(k+1)}] \geq 0$ is fulfilled for all $0 \leq i,j \leq k$. Thus, all vectors $\vec{\mathbf{p}}_j$ lie on the same side of the plane spanned by the neighbouring vectors $\vec{\mathbf{p}}_i, \vec{\mathbf{p}}_{(i+1)\mathrm{mod}(k+1)}$.

Under these assumptions, the unit vectors $(\vec{\mathbf{p}}_i)_{i=0,\dots,k}$ span the *wedge* (or, the convex cone)

$$\begin{aligned} \mathcal{W}(\vec{\mathbf{p}}_0,\dots,\vec{\mathbf{p}}_k) &= \{\vec{\mathbf{x}} \in \mathbb{R}^3 \mid \vec{\mathbf{x}} = \sum_{i=0}^{k} \lambda_i \vec{\mathbf{p}}_i \text{ with } \lambda_i \geq 0, \lambda_i \in \mathbb{R}\} \\ &= \{\vec{\mathbf{x}} \in \mathbb{R}^3 \mid [\vec{\mathbf{x}}, \vec{\mathbf{p}}_i, \vec{\mathbf{p}}_{(i+1)\mathrm{mod}(k+1)}] \geq 0 \text{ for } i = 0,\dots,k\} \end{aligned} \tag{2.10}$$

Note that the ordered sets

$$\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, \dots, \vec{\mathbf{p}}_k \rangle,\ \langle \vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2 \dots, \vec{\mathbf{p}}_k, \vec{\mathbf{p}}_0 \rangle,\ \langle \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3 \dots, \vec{\mathbf{p}}_k, \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1 \rangle,\ \dots \tag{2.11}$$

are equivalent, i.e., they span the same wedges.

Now we consider two sets of unit bounding vectors of a wedge, spanning the wedges $\mathcal{P} = \mathcal{W}(\vec{\mathbf{p}}_0, \ldots, \vec{\mathbf{p}}_k)$ and $\mathcal{Q} = \mathcal{W}(\vec{\mathbf{q}}_0, \ldots, \vec{\mathbf{q}}_l)$. We define the cross product of these wedges by

$$\mathcal{P} \times \mathcal{Q} = \{\vec{\mathbf{x}} \times \vec{\mathbf{y}} \mid \vec{\mathbf{x}} \in \mathcal{P} \text{ and } \vec{\mathbf{y}} \in \mathcal{Q}\}, \tag{2.12}$$

cf. Figure 1b. Provided this set forms again a wedge (which is the case if the vectors $\{\vec{\mathbf{p}}_i \times \vec{\mathbf{q}}_j \mid i = 0, .., k; j = 0, .., l\}$ satisfy condition (a)), then its unit bounding vectors are an appropriately ordered subset of $\{\frac{\vec{\mathbf{p}}_i \times \vec{\mathbf{q}}_j}{\|\vec{\mathbf{p}}_i \times \vec{\mathbf{q}}_j\|} \mid \ldots\}$. We denote this ordered set (which is unique up to the choice of the first vector) by

$$\langle \vec{\mathbf{r}}_0, \vec{\mathbf{r}}_1, \ldots, \vec{\mathbf{r}}_z \rangle = \langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, \ldots, \vec{\mathbf{p}}_k \rangle \times \langle \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_1, \ldots, \vec{\mathbf{q}}_l \rangle. \tag{2.13}$$

3 Sufficient convexity conditions

In the following theorem we present some criteria for the surface (2.1) which guarantee always non–negative normal curvatures at all points. The conditions described below lead to sets of inequalities for the components of the control points $\mathbf{b}_{i,j}$. However, in order to be brief we omit the complete formulation of the inequalities and present only equations of certain polynomials instead. The inequalities for the control points are easily obtained by expressing them in Bernstein–Bézier form with the help of (2.2) and using the well–known product and degree raising formulas for Bernstein polynomials,

$$B^{r_1}_{s_1}(t)\, B^{r_2}_{s_2}(t) = \frac{\binom{r_1}{s_1}\binom{r_2}{s_2}}{\binom{r_1+r_2}{s_1+s_2}}\, B^{r_1+r_2}_{s_1+s_2}(t) \tag{3.1}$$

and

$$B^r_s(t) = \frac{r+1-s}{r+1}\, B^{r+1}_s(t) + \frac{s+1}{r+1}\, B^{r+1}_{s+1}(t). \tag{3.2}$$

Theorem. *Consider a regular tensor–product Bézier surface patch (2.1). If one of the following three conditions holds, then the surface possesses always non–negative normal curvatures at all points* $\mathbf{x}(u,v)$ *with parameter values* $(u,v) \in [0,1]^2$.

(i) *(Conditions of degree* 6*) The Bézier coefficients of the three bivariate polynomials*

$$\begin{aligned} L_1(u,v) &= [\mathbf{x}_{uu}(u,v), \mathbf{x}_u(.), \mathbf{x}_v(.)]\; [\mathbf{x}_{vv}(.), \mathbf{x}_u(.), \mathbf{x}_v(.)] \\ &\qquad - [\mathbf{x}_{uv}(.), \mathbf{x}_u(.), \mathbf{x}_v(.)]^2, \\ L_2(u,v) &= [\mathbf{x}_{uu}(u,v), \mathbf{x}_u(u,v), \mathbf{x}_v(u,v)] \quad \textit{and} \\ L_3(u,v) &= [\mathbf{x}_{vv}(u,v), \mathbf{x}_u(u,v), \mathbf{x}_v(u,v)] \end{aligned} \tag{3.3}$$

in u, v *are non–negative. The Bézier coefficients of the first polynomial* $L_1(u,v)$ *lead to a system of* $(6m-3)(6n-3)$ *inequalities of degree* 6 *in the control points* $\mathbf{b}_{i,j}$*, whereas the coefficients of the polynomials* L_2 *and* L_3 *yield systems of* $(3m-2)(3n)$ *and* $(3m)(3n-2)$ *cubic inequalities.*

(ii) *(Cubic conditions) Choose two appropriate finite sequences* $(s_i)_{i=0,\ldots,p}$, $(t_j)_{j=0,\ldots,q}$ *of real numbers with* $p, q \geq 2$ *and*

$$0 = s_0 < s_1 < \ldots < s_p = 1, \quad 0 = t_0 < t_1 < \ldots < t_q = 1. \tag{3.4}$$

The Bézier coefficients of the $p+q$ bivariate polynomials

$$\begin{aligned} S_{i+1}(u,v) = \Big[& (1-s_i)(1-s_{i+1})\mathbf{x}_{uu}(u,v) \\ & + (s_i+s_{i+1}-2\,s_i\,s_{i+1})\mathbf{x}_{uv}(.) + s_i\,s_{i+1}\mathbf{x}_{vv}(.)\,,\,\mathbf{x}_u(.)\,,\,\mathbf{x}_v(.)\Big] \quad \textit{and} \\ T_{j+1}(u,v) = \Big[& (1-t_j)(1-t_{j+1})\mathbf{x}_{uu}(u,v) \\ & - (t_j+t_{j+1}-2\,t_j\,t_{j+1})\mathbf{x}_{uv}(.) + t_j\,t_{j+1}\mathbf{x}_{vv}(.)\,,\,\mathbf{x}_u(.)\,,\,\mathbf{x}_v(.)\Big] \\ & (i=0,\ldots,p-1;\ j=0,\ldots,q-1) \end{aligned} \tag{3.5}$$

in u,v are non–negative. This leads to a system of $9\,m\,n\,(p+q)$ cubic inequalities for the components of the control points of the surface.

(*iii*) *(Linear conditions) Choose two appropriate ordered sets $\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, ..., \vec{\mathbf{p}}_k \rangle$ and $\langle \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_1, ..., \vec{\mathbf{q}}_l \rangle$ of unit bounding vectors of two wedges. The cross product of the two wedges exists and its bounding vectors are denoted by $\langle \vec{\mathbf{r}}_0, \vec{\mathbf{r}}_1, \ldots, \vec{\mathbf{r}}_z \rangle$, cf. (2.13). In addition, consider again the real numbers s_i, t_j as introduced in (ii).*

The Bézier coefficients of the $(k+1)+(l+1)$ bivariate polynomials

$$\begin{aligned} P_i(u,v) &= [\mathbf{x}_u(u,v), \vec{\mathbf{p}}_i, \vec{\mathbf{p}}_{(i+1)\mathrm{mod}(k+1)}] \quad i=0,\ldots,k \\ Q_j(u,v) &= [\mathbf{x}_v(u,v), \vec{\mathbf{q}}_j, \vec{\mathbf{q}}_{(j+1)\mathrm{mod}(l+1)}] \quad j=0,\ldots,l \end{aligned} \tag{3.6}$$

in u,v are non–negative. Moreover, the Bézier coefficients of the $(z+1)\ (p+q)$ bivariate polynomials

$$\begin{aligned} \hat{S}_{h,i+1}(u,v) = \Big(& (1-s_i)(1-s_{i+1})\mathbf{x}_{uu}(u,v) \\ & + (s_i+s_{i+1}-2\,s_i\,s_{i+1})\mathbf{x}_{uv}(.) + s_i\,s_{i+1}\mathbf{x}_{vv}(.)\Big) \circ \vec{\mathbf{r}}_h \quad \textit{and} \\ \hat{T}_{h,j+1}(u,v) = \Big(& (1-t_j)(1-t_{j+1})\mathbf{x}_{uu}(u,v) \\ & - (t_j+t_{j+1}-2\,t_j\,t_{j+1})\mathbf{x}_{uv}(.) + t_j\,t_{j+1}\mathbf{x}_{vv}(.)\Big) \circ \vec{\mathbf{r}}_h \\ & (i=0,\ldots,p-1;\ j=0,\ldots,q-1;\ h=0,\ldots,z) \end{aligned} \tag{3.7}$$

in u,v are non–negative too. The Bézier coefficients of the polynomials P_i and Q_j leads to systems of $m(n+1)(k+1)$ and $n(m+1)(l+1)$ inequalities, whereas the two polynomials in (3.7) yield another system which consists of $(m+1)(n+1)(z+1)(p+q)$ inequalities. All inequalities are linear in the components of control points of the surface (2.1).

Proof. (*i*) Resulting from the non–negativity of their Bézier coefficients, the polynomials $L_1(u,v), L_2(u,v), L_3(u,v)$ have non–negative values for all $0 \le u,v \le 1$. As

$$L_1 = \det \begin{pmatrix} [\mathbf{x}_{uu}, \mathbf{x}_u, \mathbf{x}_v] & [\mathbf{x}_{uv}, \mathbf{x}_u, \mathbf{x}_v] \\ [\mathbf{x}_{uv}, \mathbf{x}_u, \mathbf{x}_v] & [\mathbf{x}_{vv}, \mathbf{x}_u, \mathbf{x}_v] \end{pmatrix} \ge 0 \tag{3.8}$$

holds, both eigenvalues of the 2×2 matrix are either non–negative or non–positive. This implies together with $L_2 \ge 0$ and $L_3 \ge 0$, that the quadratic form (2.8) is non–negative definite.

(*ii*) Again, due to the non–negativity of their Bézier coefficients, the polynomials $S_{i+1}(u,v), T_{j+1}(u,v)$ have non–negative values for all $0 \le u,v \le 1$. We consider an arbitrary fixed point $(u,v) \in [0,1]^2$. The quadratic form (2.8) is homogeneous in ξ and η, therefore it suffices to consider only the tangent directions (2.6) with $(\xi,\eta) = (\xi, 1-\xi)$ or $(\xi,\eta) = (\xi, -1+\xi)$

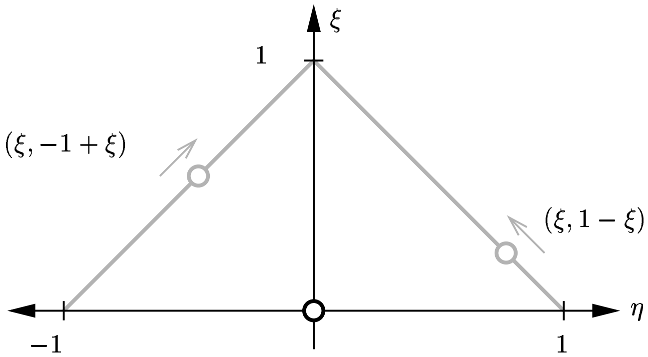

FIGURE 2. Tangent directions $(\xi, 1-\xi)$ and $(\xi, -1+\xi)$ with $0 \le \xi \le 1$.

and $0 \le \xi \le 1$, see Figure 2. Consider the first set of tangents. By substituting $\eta = 1-\xi$, the quadratic form (2.8) is transformed into

$$Q(\xi) = \left[B_0^2(\xi)\,\mathbf{x}_{uu} + B_1^2(\xi)\,\mathbf{x}_{uv} + B_2^2(\xi)\,\mathbf{x}_{vv}\,,\,\mathbf{x}_u\,,\,\mathbf{x}_v\right] \tag{3.9}$$

with the quadratic Bernstein polynomials $B_i^2(\xi)$. For any $\xi \in [0,1]$ values s_j, s_{j+1} with $s_j \le \xi < s_{j+1}$ exist. Computing the Bézier representation of the polynomial $Q(\xi)$ with respect to the interval $[s_j, s_{j+1}] \subset [0,1]$ yields

$$Q(\xi) = C_0\; B_0^2(\frac{\xi - s_j}{s_{j+1} - s_j}) + C_1\; B_1^2(\frac{\xi - s_j}{s_{j+1} - s_j}) + C_2\; B_2^2(\frac{\xi - s_j}{s_{j+1} - s_j}) \tag{3.10}$$

with the three coefficients

$$C_0 = \left[(1-s_j)^2\,\mathbf{x}_{uu} + 2\,s_j\,(1-s_j)\,\mathbf{x}_{uv} + s_j^2\,\mathbf{x}_{vv},\ \mathbf{x}_u, \mathbf{x}_v\right], \tag{3.11}$$

$$C_1 = [(1-s_j)(1-s_{j+1})\,\mathbf{x}_{uu} + (s_j + s_{j+1} - 2\,s_j\,s_{j+1})\,\mathbf{x}_{uv} + s_j\,s_{j+1}\,\mathbf{x}_{vv},\ \mathbf{x}_u, \mathbf{x}_v] \tag{3.12}$$

and

$$C_2 = \left[(1-s_{j+1})^2\,\mathbf{x}_{uu} + 2\,s_{j+1}(1-s_{j+1})\,\mathbf{x}_{uv} + s_{j+1}^2\,\mathbf{x}_{vv},\ \mathbf{x}_u, \mathbf{x}_v\right]. \tag{3.13}$$

The coefficients can be generated with the help of the blossom of the polynomial $Q(\xi)$, see [7].

The values of the Bernstein polynomials $B_i^2(\frac{\xi - s_{j+1}}{s_{j+1} - s_j})$ are non–negative as $s_j \le \xi < s_{j+1}$ holds. The middle coefficient C_1 is guaranteed to be non–negative, due to $S_{j+1}(u,v) \ge 0$.

Assume $j \ge 1$. Then we have both $S_j(u,v) \ge 0$ and $S_{j+1}(u,v) \ge 0$ which leads to

$$[(1-s_j)(1-\hat{s})\mathbf{x}_{uu} + (s_j + \hat{s} - 2 s_j \hat{s})\mathbf{x}_{uv} + s_j \hat{s}\,\mathbf{x}_{vv},\ \mathbf{x}_u, \mathbf{x}_v] \ge 0 \tag{3.14}$$

for $\hat{s} = s_{j-1}$ (from $S_j(u,v) \ge 0$) and for $\hat{s} = s_{j+1}$ (from $S_{j+1}(u,v) \ge 0$). Moreover this expression depends linearly on $\hat{s}$, hence the inequality is also true for $\hat{s} = s_j$. Resulting from this observation, the coefficient C_0 is non–negative for $j \ge 1$.

For $j = 0$ we get from $S_1(u,v) \ge 0$ and $T_1(u,v) \ge 0$ the inequalities

$$[(1-s_1)\mathbf{x}_{uu} + s_1\mathbf{x}_{uv}\,,\,\mathbf{x}_u, \mathbf{x}_v] \ge 0 \text{ and } [(1-t_1)\mathbf{x}_{uu} - t_1\mathbf{x}_{uv}\,,\,\mathbf{x}_u, \mathbf{x}_v] \ge 0 \tag{3.15}$$

(Note $s_0 = t_0 = 0$). Due to $p, q \ge 2$ we have $s_1, t_1 < 1$, thus we see that the coefficient C_0 (which is then equal to $[\mathbf{x}_{uu}, \mathbf{x}_u, \mathbf{x}_v]$) is non–negative also for $j = 0$. Similarly it can be shown that the third coefficient C_2 is always non–negative. Therefore the quadratic form (2.8) has non–negative values for all tangent directions (2.6) with $(\xi, \eta) = (\xi, 1-\xi)$

and $0 \le \xi \le 1$. Analogous considerations (based on the polynomials $T_{j+1}(u,v)$) prove this assertion for tangent directions with $(\xi, \eta) = (\xi, -1+\xi)$ and $0 \le \xi \le 1$.

(*iii*) Once more, the bivariate polynomials $P_i(u,v)$, $Q_j(u,v)$, $\hat{S}_{h,i+1}(u,v)$ and $\hat{T}_{h,j+1}(u,v)$ are guaranteed to have non–negative values for all $0 \le u, v \le 1$, resulting from the non–negativity of their Bézier coefficients. We consider an arbitrary, but fixed point $\mathbf{x}(u,v)$ with parameters $(u,v) \in [0,1]^2$. According to $P_i(u,v) \ge 0$ and $Q_j(u,v) \ge 0$, the first partial derivative vectors $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ are contained in the wedges spanned by $\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, \dots, \vec{\mathbf{p}}_k \rangle$ and $\langle \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_l \rangle$, respectively, see (2.10). Thus, their cross product $\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)$ is contained in the wedge spanned by $\langle \vec{\mathbf{r}}_0, \vec{\mathbf{r}}_1, \dots, \vec{\mathbf{r}}_z \rangle$. Therefore we can find real coefficients $\zeta_0, \dots, \zeta_z \ge 0$ such that

$$\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) = \sum_{h=0}^{z} \zeta_h \, \vec{\mathbf{r}}_h \tag{3.16}$$

holds. From $\hat{S}_{h,i+1}(u,v) \ge 0$ and using the definition of the bracket product $[.,.,.]$ we conclude that the polynomials $S_{i+1}(u,v)$ from (3.5) fulfill the inequalities

$$\begin{aligned} S_{i+1}(u,v) &= \Big((1-s_i)(1-s_{i+1})\mathbf{x}_{uu}(u,v) + (s_i+s_{i+1}-2\,s_i\,s_{i+1})\mathbf{x}_{uv}(.) \\ &\qquad + s_i\,s_{i+1}\mathbf{x}_{vv}(.) \Big) \circ \sum_{h=0}^{z} \zeta_h \, \vec{\mathbf{r}}_h \\ &= \sum_{h=0}^{z} \zeta_h \, \hat{S}_{h,i+1}(u,v) \\ &\ge 0 \qquad\qquad i = 0, \dots, p-1. \end{aligned} \tag{3.17}$$

Analogously we obtain $T_{j+1}(u,v) \ge 0$ $(j = 0, \dots, q-1)$. Now we use again the arguments presented previously in (*ii*). So we conclude that the quadratic form (2.8) is non–negative definite for all points $\mathbf{x}(u,v)$. This completes the proof. □

Whereas the convexity conditions (*i*) lead to a system of inequalities of degree six in the components of the control points, the inequalities which result from (*ii*) are cubic and those obtained from (*iii*) are only linear. The latter inequalities are much better suited for using them in an optimization process. Together with an appropriate objective function they lead to linear or quadratic programming problems which can be solved with the help of the powerful tools from optimization theory. This will be illustrated by discussing the lifting problem in the next section. In contrast with this, using the cubic or even the degree 6 inequalities will always lead to a nonlinear problem.

However, the linear and cubic conditions require some constants which have to be chosen. This can be done automatically with the help of a *reference surface*. For instance, the initial surface of a modification may serve as such a reference surface. The set of unit bounding vectors of a wedge $\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, \dots, \vec{\mathbf{p}}_k \rangle$ and $\langle \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_1, \dots, \vec{\mathbf{q}}_l \rangle$ are found from the wedges spanned by the control points of the first partial derivatives. These wedges are enlarged in order to provide enough degrees of freedom for the optimization. The real numbers $(s_i)_{i=0,\dots,p}$ and $(t_j)_{j=0,\dots,q}$ result from an adaptive refinement of the Bézier representations of the polynomials $Q(\xi)$ (which appear in the proof of (*ii*)) and the corresponding polynomials for the second set of tangent directions.

Any surface (2.1) with strictly positive Gaussian curvature can be subdivided into a finite number of subsegments, such that the convexity can be guaranteed with the help of the conditions obtained from the above theorem. This results immediately from the fact, that iterated subdivision implies convergence of the control polygons to the derivatives. Moreover, if we increase the number of the knots $(s_i)_{i=0,\dots,p}$ and $(t_j)_{j=0,\dots,q}$ and distribute them uniformly over the unit interval $[0,1]$, then the Bézier coefficients of the polynomials $S_{i+1}(u,v)$ and $T_{j+1}(u,v)$ converge to the values of the quadratic form (2.8).

The approach presented in the theorem for generating linear convexity conditions in some sense decouples the first from the second partial derivatives. More precisely, both derivatives are bounded by certain wedges: the first derivatives by those described by the inequalities obtained from the Bézier coefficients of $P_i(u,v)$ and $Q_j(u,v)$, and the second (directional) derivatives by the the wedges described by the inequalities obtained from the Bézier coefficients of $\hat{S}_{h,i+1}(u,v)$ and $\hat{T}_{h,j+1}(u,v)$. As the main advantage, any computation for these wedges (e.g., convex hull computations) can be reduced to a computation with planar polygons.

As a different approach one could also try to bound the control polygons of the derivatives by appropriate polyhedrons. But computations for polyhedrons in 3–space are much more complicated. For instance, the convex hull of a planar point set can easily be found, whereas the analogous problem in 3–space is more difficult. So the presented approach seems to be the more natural one.

Of course, it is possible to apply the above theorem to tensor–product Bézier functions $f(u,v)$, i.e., to bivariate polynomials, simply by choosing the parametric representation $\mathbf{x}(u,v) = (\,u\;\; v\;\; f(u,v)\,)^\top$. Then the conditions obtained from the first part (i) form a system of quadratic inequalities, whereas those obtained from the second part (ii) are even only linear in the Bézier coefficients.

4 An application: Lifting of Bézier surfaces

In this section we illustrate the convexity conditions of the theorem by an application. Let a convex surface patch, e.g., a tensor–product Bézier patch (2.1), be given. This surface is to be modified, by pulling one ore more points upwards or pushing them downwards, This modification is to preserve the convexity of the surface. Moreover, certain boundary conditions are to be satisfied (e.g., the first partial derivatives along the boundaries are to be kept).

This kind of problems, called the "lifting" of a surface, may arise in the design process of car body surfaces or also ship hulls. It has been discussed by Schichtel in his Ph.D. thesis [12] using a different approach, see also [13]. Schichtel introduces an auxiliary surface which is non–convex in general. The auxiliary surface fulfills the boundary conditions and its shape reflects the intended result of the lifting. The one–parameter set of convex combinations of the original surface and the auxiliary surface is examined and the subset of convex surfaces is identified. In this step one has to solve a nonlinear programming problem. Schichtel proposes to use a simple gradient method. According to the magnitude of the desired magnification, a surface from this subset of convex surfaces is chosen as the result of the lifting procedure. As a drawback of Schichtel's method, the result depends on the choice of the auxiliary surface.

4.1 The method

We present another algorithm which is based on the previously derived convexity conditions.

1.) Choose the initial surface (which is assumed to be strongly convex) as a reference surface and generate linear sufficient convexity conditions, based on part (iii) of the above theorem. For generating the constraints we have to subdivide twice. At first, the surface patch is subdivided into smaller patches, with the help of the de Casteljau scheme. The Bézier coefficients of the sub–patches are certain constant affine combinations of the original coefficients. This subdivision stops,

- if bounding wedges $\langle \vec{\mathbf{p}}_0, \vec{\mathbf{p}}_1, ..., \vec{\mathbf{p}}_k \rangle$ and $\langle \vec{\mathbf{q}}_0, \vec{\mathbf{q}}_1, ..., \vec{\mathbf{q}}_l \rangle$ for the control polygons of the first derivatives can be found such that the cross–product $\langle \vec{\mathbf{r}}_0, \vec{\mathbf{r}}_1, \dots, \vec{\mathbf{r}}_z \rangle$ of these wedges exists, and
- if the Bézier coefficients with respect to (u, v) of the $z+1$ bivariate polynomials

$$\big(B^2_0(\xi)\ \mathbf{x}_{uu}(u,v) \pm B^2_1(\xi)\ \mathbf{x}_{uv}(.) + B^2_2(\xi)\ \mathbf{x}_{vv}(.) \big) \circ \mathbf{r}_i \quad (i = 0, .., z) \tag{4.1}$$

  possess strictly positive values for all $\xi \in [0, 1]$. For fixed ξ, the expression (4.1) is a bivariate polynomial of degree (m, n) which depends on the surface parameters u and v. The polynomials with the $+$ and $-$ sign arise from the of the tangent directions $(\xi, 1-\xi)$ and $(\xi, -1+\xi)$, respectively.

The bounding wedges for the first derivative vectors are slightly enlarged in order to ensure that the subsequent optimization finds enough degrees of freedom at its initial point.

In the second step, the polynomials (4.1) are subdivided with respect to ξ until all resulting Bézier coefficients with respect to ξ, u and v are positive. This subdivision produces the two knot sequences $(s_i)_{i=0,\dots,p}$ and $(t_j)_{j=0,\dots,q}$.

2.) Choose an appropriate objective function. For instance, if the surface point $\mathbf{x}(0.5, 0.5)$ is to be pulled into the direction $\vec{\mathbf{l}}$, then one may choose the objective function

$$||\mathbf{x}(0.5, 0.5) - (\mathbf{x}_{\text{initial}}(0.5, 0.5) + \vec{\mathbf{l}})||^2 + \lambda \text{ (regularizing term)}. \tag{4.2}$$

The vector $\vec{\mathbf{l}} \in \mathbb{R}^3$ is called the *lift vector.* A similar objective function can be used if more than one surface point is to be lifted, as depicted in [12, p. 73]. The regularizing term has been introduced in order to guarantee that the gradient of the objective function possesses maximal rank. For example, one may use the sum of the squared lengths of the control net,

$$\sum_{i=0}^{m} \sum_{j=0}^{n-1} ||\mathbf{b}_{i,j+1} - \mathbf{b}_{i,j}||^2 + \sum_{i=0}^{m-1} \sum_{j=0}^{n} ||\mathbf{b}_{i+1,j} - \mathbf{b}_{i,j}||^2 . \tag{4.3}$$

The weight $\lambda \in \mathbb{R}$ is chosen such that the regularizing term contributes very little to the value of the objective function, compared with the first part.

The objective function (4.2) is a quadratic expression in the components of the control points $\mathbf{b}_{i,j}$.

3.) The control points of the modified surface are computed by minimizing the objective function (4.2) subject to the linearized convexity conditions obtained from 1.) and to boundary conditions. For instance, in the case of C^1 boundary conditions we keep the first two rows of boundary control points of the surface. This quadratic programming problem (i.e., minimization of a quadratic objective function subject to linear equality and inequality constraints) can be solved exactly in finite time (at least theoretically), see e.g. the textbook by Fletcher [4]. Our implementation is based on an active set strategy as described in [4]. As an alternative one may use an interior point method like that of the LOQO package by Vanderbei [14]. In our examples, however, we obtained better results with the help of the active set strategy. This method generalizes the simplex algorithm to the case of a quadratic objective function. The reference surface serves as initial point for the optimization. In order to avoid degenerate situations it is of crucial importance to detect and to remove dependencies of the set of constraints as far as possible.

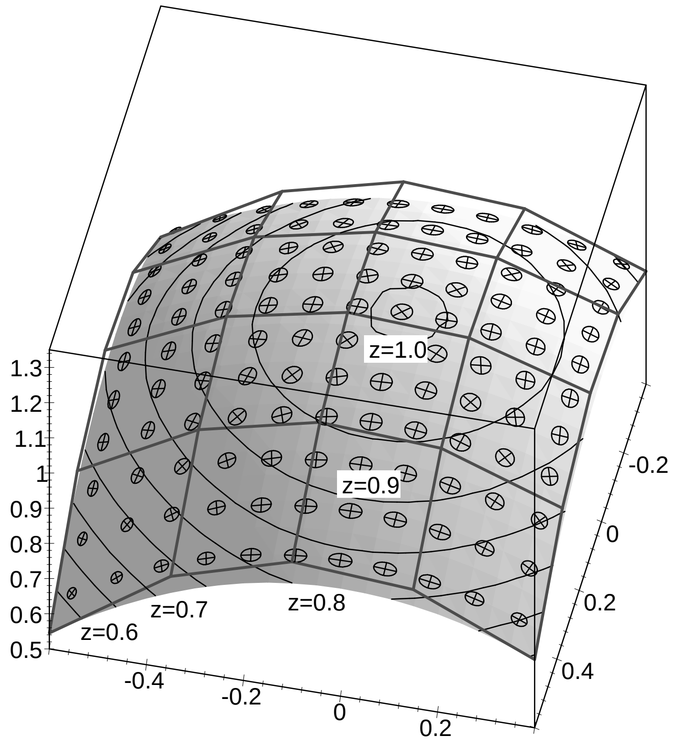

FIGURE 3. Convexity–preserving lifting of Bézier surfaces:the original surface.

4.) Check, whether a sufficiently large modification of the surface has been achieved. If not, then one may iterate the whole construction by choosing the result of the optimization as a new initial surface (→ 1.). One should also check whether the surface possesses enough degrees of freedom, such that the intended modification can be achieved.

On the other hand, if the modification is too big, then one may consider the one–parameter set of convex combinations of the initial surface and the modified surface. All these convex combinations are guaranteed to be convex as they fulfill the linearized convexity conditions. One of them is chosen as result of the lifting procedure, according to the magnitude of the intended modification.

4.2 An example

In our example, the initial surface is chosen as a biquartic tensor–product Bézier surface patch. Figure 3 shows the original surface and its control polygon. The curvature is visualized by some ellipses in the tangent planes of the surface. The principal diameters of the ellipses indicate the principal curvature directions, their length is proportional to the corresponding principal curvatures (which are the extreme values of the normal curvatures at this point). Hence, the area of the ellipses is proportional to the Gaussian curvature. Note that these ellipses are not the Dupin indicatrices of the surfaces; these indicatrices would result by choosing the length of the diameters proportional to the principal curvature radii (but this causes problems for the visualization if the curvatures are rather small).

Moreover the Figure also shows some contour lines $z =$ constant of the surface (stepsize 0.05). Note that the surface is not symmetric! The Gaussian curvature $K(u, v)$ has been

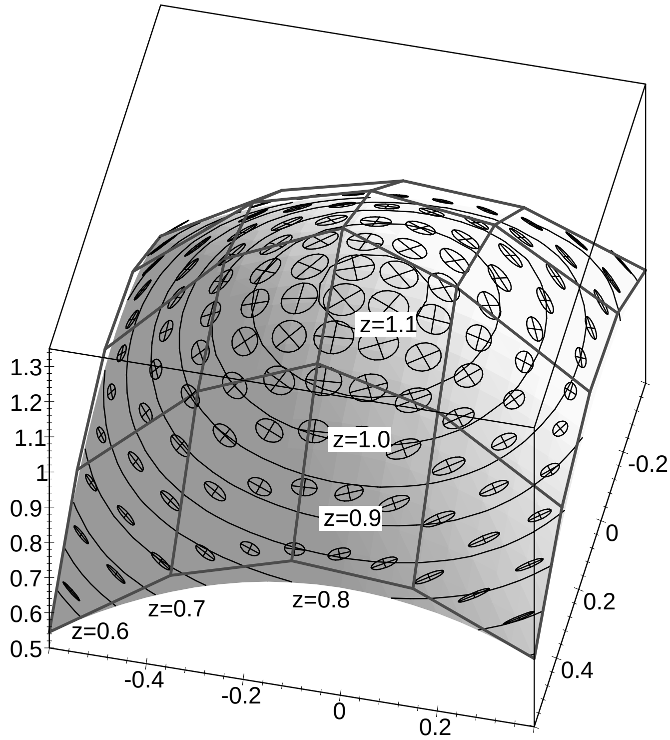

FIGURE 4. The surface after lifting with lift vector $(0\ 0\ 5)^\top$ subject to C^0 boundary conditions.

plotted in Figure 6a. Its values vary between 0.247 and 1.172.

At first we modify the surface by pulling its point $\mathbf{x}(0.5, 0.5)$ upwards, whereby the boundary curves (i.e., the boundary control points) are kept. Using an automatic procedure as outlined in 1.) we find linear sufficient convexity conditions according to part (*iii*) of the theorem. This leads to a system of 472 inequalities for the 27 unknown components of the inner surface control points.

The control points of the modified surface are found by minimizing the objective function (4.2) subject to convexity and boundary conditions. The lift vector $\vec{\mathbf{l}} = (\,0\ \ 0\ \ 5\,)^\top$ indicates the direction of the desired modification.

In the first iteration of the above–described method, the surface point $\mathbf{x}(0.5, 0.5)$ is moved from $(\,0.1\ -0.1\ 0.992\,)^\top$ to $(\,0.105\ -0.105\ 1.096\,)^\top$. The Gaussian curvature now varies between 0.210 and 3.999.

In order to improve the obtained result we iterate the whole procedure. The modified surface serves as a new initial surface patch. For finding linear convexity conditions according to part (*iii*) of the theorem, the surface patch now has to be subdivided into 7 segments. We obtain a system of 5942 inequalities for the 27 unknowns. Minimizing (4.2) subject to these constraints yields the surface which is depicted by Figure 4. The point $\mathbf{x}(0.5, 0.5)$ is now at $(\,0.105\ -0.105\ 1.108\,)$. The Gaussian curvature (see Figure 6b) varies between 0.08 and 4.426, its minimum occurs in one of the corners where the surface has an "almost parabolic" point. The ellipses in Figure 4 clearly visualize the modified curvature distribution. Also the Gaussian curvature plot in Figure 6b shows some significant changes.

In a second experiment we try to push the surface downwards. The degree of the

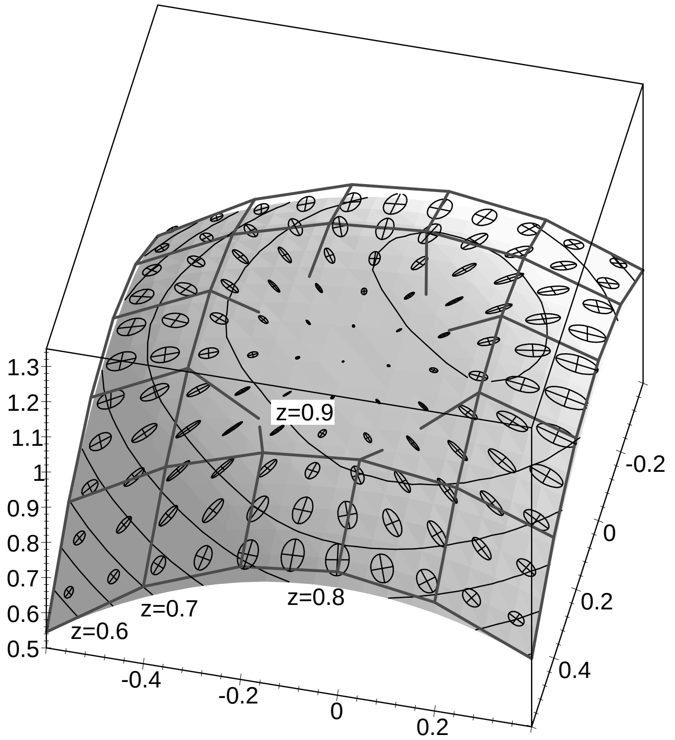

FIGURE 5. The surface after lifting with lift vector $(0\ 0\ -5)^\top$ subject to C^1 boundary conditions.

initial surface patch is raised to $(5,5)$ first. The modified surface is found be minimizing the objective function (4.2) with lift vector $(0\ 0\ -5)^\top$ subject to C^1 boundary conditions (i.e., the first two rows of boundary control points remain unchanged) and convexity constraints. The generation of linear sufficient convexity conditions is iterated three times and leads to quadratic programming problems with 240, 2902 and 10 080 linear inequalities and 12 unknowns. The result is shown in Figure 5. The modified surface point $\mathbf{x}(0.5, 0.5)$ is now at $(\,0.101\ -0.102\ 0.928\,)^\top$. The Gaussian curvature of the modified surface (see Figure 6c) varies between 0.004 and 2.535, with the minimum in the central region of the surface. The modified surface has almost flat points in this region. The new curvature distribution is visualized by the ellipses in Figure 5 and can also be seen from the Gaussian curvature plot in Figure 6c.

The computing time in these examples was in the order of a few minutes. As observed in the experiments, if the surface is close to its extreme shape (indicated by surface regions with $K \approx 0$), then the number of inequalities explodes. One should therefore restrict the number of permitted subdivisions. The biggest modification of the shape is often achieved in the first step of the iteration, whereby the number of inequalities is relatively small.

5 Concluding remarks

In this article we described a method for the generation of linear constraints which guarantee the convexity of parametric tensor–product Bézier surfaces. This method can easily be generalized to tensor–product B-spline surfaces. Instead of the simple formulas (3.1) and (3.2) one has to use the analogous identities for B-splines, see [10]. As an alterna-

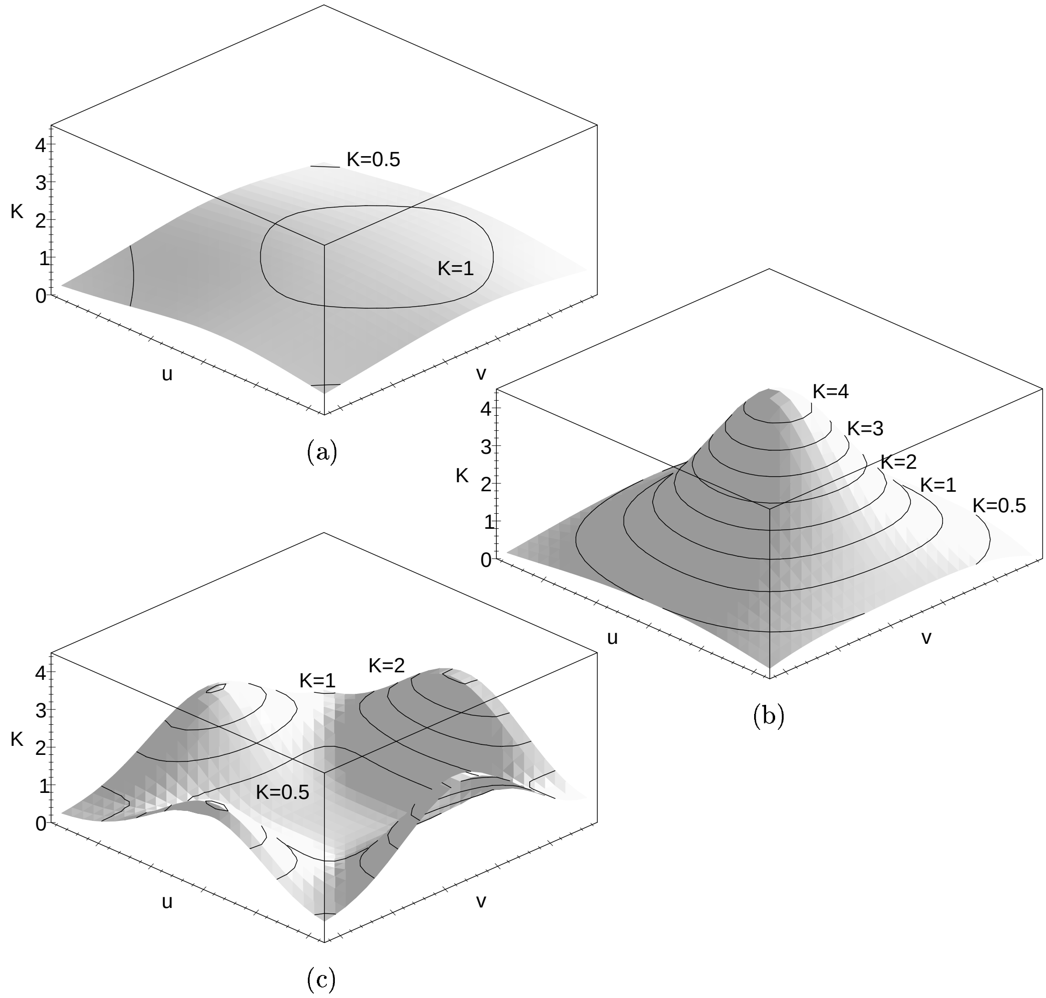

FIGURE 6. The Gaussian curvature of the original surface (a), after lifting it upwards (b), and after lifting it downwards (c).

tive one may represent a B-spline surface as a collection of Bézier surfaces and apply the above-described methods to the single patches.

With the help of the linearized constraints it is possible to formulate several tasks of surface construction or modification as a sequence of optimization problems with linear constraints. This has been illustrated by the convexity–preserving lifting of parametric Bézier surfaces.

A similar approach to shape preserving approximation by planar B–spline curves has been derived in [8]. That article generalizes an algorithm proposed by Dierckx [2, 3] to the case of planar parametric curves.

Based on linear convexity conditions for spline functions, an algorithm for least–squares approximation of functional data by tensor–product spline functions subject to piecewise convexity/concavity conditions has been developed in [9]. Unlike the functional case, linearized convexity conditions for parametric surfaces require a reference surface which specifies the expected shape. Moreover, convex spline functions form a convex set, whereas convex parametric spline surfaces do not. In order to develop an algorithm for shape–preserving approximation by parametric surfaces one has to find a construction of an appropriate reference surface at first. For instance, one might choose the result of the functional approximation as such a surface. This, however, will fail for non–functional data.

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