Rational Patches on Quadric Surfaces

Roland Dietz, Josef Hoschek and Bert Jüttler

November 25, 2003

Authors' address

Prof. Dr. J. Hoschek, R. Dietz and B. Jüttler
Technische Hochschule Darmstadt
Fachbereich Mathematik, AG 3

Schloßgartenstraße 7

D – 64289 Darmstadt

Germany

Electronic Mail: hoschek@mathematik.th-darmstadt.de

Abstract

The present paper discusses rational curve segments and surface patches on quadric surfaces. Detailed constructions of rational Bézier patches from given boundaries on the unit sphere and on the hyperbolic paraboloid are presented based on a generalization of the stereographic projection. The method is applied to the interpolation with rational curves on quadrics. The results are extended to rational B-spline representations by discussing products of B-spline functions. Finally, the generalization of the constructions to arbitrary nondegenerated quadric surfaces is outlined.

Keywords. Rational Bézier and B-spline curves and surfaces, biquadratic Bézier patch, quadric surfaces, interpolation on quadrics, generalized stereographic projection, products of B-spline functions

Introduction

Most of Computer-Aided Design systems describe surface patches by polynomial or rational parametric representations. As an important advantage, rational representations (e.g. NURBS-curves and -surfaces) support the exact representation of quadric surfaces (like spheres, ellipsoides, hyperboloids of one or two sheets, elliptic and hyperbolic paraboloids) which traditionally play an important role in industrial applications. Several authors have developed different constructions of rational patches (especially of rational Bézier surface patches) on quadric surfaces [1-7]. The present paper is based on an algebraic approach introduced in Hoschek & Seemann '92 [8]. That approach has led to a generalization of the stereographic projection on quadric surfaces. New results concerning biquadratic Bézier surface patches on quadrics and surface patches on quadrics whose boundaries are conic sections have been derived with help of this generalized stereographic projection in Dietz et al.'93 [9]. Additionally, interpolation of given points with rational curves on the sphere has been shown to be a linear problem.

This paper applies the above results and presents detailed constructions of rational Bézier and B-spline surface patches on quadric surfaces. First of all, some notations from projective geometry are introduced. Section 2 and 3 discuss rational patches on the unit sphere and on the hyperbolic paraboloid, respectively. Section 4 deals with rational B-spline representations on quadrics. Finally, section 5 extends the obtained results to arbitrary quadrics.

1 Notations

The scene of the following considerations is the projectively closed three-dimensional real Euclidean space $\bar{E}^3$. Its points ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ...) and planes ($\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$, ...) are described by *homogeneous coordinate vectors* from $\mathbb{R}^4$ (see e.g. Coxeter'64 [10]). The point $\mathbf{a}$ lies on the plane $\hat{\mathbf{b}}$ iff $\langle \mathbf{a}, \hat{\mathbf{b}} \rangle = 0$ holds. (The symbol $\langle \cdot, \cdot \rangle$ denotes the usual inner product of vectors.)

The cartesian coordinate vectors of (finite) points are $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, $\underline{\mathbf{c}}$, ... They result from dividing by the 0-th components:

$$\underline{\mathbf{p}} = \frac{1}{p_0} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \text{where} \quad \mathbf{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} . \qquad (1)$$

The lines $(1 \quad \lambda \quad 0 \quad 0)^\top$, $(1 \quad 0 \quad \mu \quad 0)^\top$ and $(1 \quad 0 \quad 0 \quad \nu)^\top$ ($\lambda, \mu, \nu \in \mathbb{R}$) are the x- , y- and z- axis of the coordinate system, respectively.

Consider a point $\mathbf{p} = (p_0\, p_1\, p_2\, p_3\,)^\top$. The abbreviations

$$
\begin{array}{lclllll}
\hat{\mathbf{p}} & = & (& p_1 & -p_0 & -p_3 & p_2 &)^\top, \\
\mathbf{p}^\perp & = & (& -p_3 & p_2 & -p_1 & p_0 &)^\top, \\
\hat{\mathbf{p}}^\perp & = & (& -p_2 & -p_3 & p_0 & p_1 &)^\top, \\
\mathbf{p}^* & = & (& p_0 & p_1 & 0 & 0 &)^\top, \\
\mathbf{p}^{**} & = & (& 0 & 0 & p_2 & p_3 &)^\top, \\
\hat{\mathbf{p}}^* & = & (& p_1 & -p_0 & 0 & 0 &)^\top \quad \text{and} \\
\hat{\mathbf{p}}^{**} & = & (& 0 & 0 & -p_3 & p_2 &)^\top
\end{array}
\tag{2}
$$

are introduced in order to simplify notations.
Let $B \neq O$ be a symmetric $(4,4)$-matrix. The set of all points $\mathbf{x}$ satisfying $\mathbf{x}^\top B\, \mathbf{x} = 0$ forms a *quadric*. The equations $\mathbf{x}^\top U\, \mathbf{x} = 0$ and $\mathbf{x}^\top H\, \mathbf{x} = 0$ with

$$
U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\tag{3}
$$

describe the unit sphere and the (normalized) hyperbolic paraboloid, respectively.
The symbols $\hat{\mathbf{p}} = \mathbf{a} \vee \mathbf{b} \vee \mathbf{c}$ and $\mathbf{p} = \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \wedge \hat{\mathbf{c}}$ denote the plane spanned by three points and the intersection point of three planes, respectively. The coordinates of $\hat{\mathbf{p}}$ resp. $\mathbf{p}$ are

$$
p_i = (-1)^i \det \begin{pmatrix} a_j & a_k & a_l \\ b_j & b_k & b_l \\ c_j & c_k & c_l \end{pmatrix}
\tag{4}
$$

where $(i,j,k,l) \in \{(0,1,2,3,),(1,2,3,0),(2,3,0,1),(3,0,1,2)\}$.

The use of homogeneous coordinates allows a very compact description of rational curve segments and surface patches: For example, a rational Bézier surface patch of degree (m,n) is given by

$$
\mathbf{x}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u)\, B_j^n(v)\, \mathbf{b}_{i.j} \quad ((u,v) \in [0,1]^2)
\tag{5}
$$

with the Bernstein polynomials $B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}$ and the homogeneous control points $\mathbf{b}_{i.j}$ (see e.g. [11]). A rational B-spline surface patch results from (5) by replacing the Bernstein polynomials with B-spline basis functions $N_{i.k}(t)$.

2 Rational patches on the unit sphere

This section discusses rational patches on the unit sphere $\mathbf{x}^\top U\mathbf{x} = 0$. The set of all points of this sphere is denoted by U. The letter P means the set of all points of the equator plane of the unit sphere ($p_3 = 0$).

2.1 The generalized stereographic projection

The stereographic projection is a standard method for the construction of surface patches on quadric surfaces (see e.g. Geise & Langbecker'90 [4]). Let $\mathbf{z}$ denote the point $(1\,0\,0\,1)^\top$, i.e. the "north pole" of the unit sphere. The line connecting an arbitrary point $\mathbf{p} \in P$ with $\mathbf{z}$ intersects the unit sphere in exactly two points: in $\mathbf{z}$ and in a second one $\sigma(\mathbf{p})$ where

$$\sigma(\mathbf{p}) = \begin{pmatrix} p_0^2 + p_1^2 + p_2^2 \\ 2p_0p_1 \\ 2p_0p_2 \\ p_1^2 + p_2^2 - p_0^2 \end{pmatrix}. \tag{6}$$

The map $\sigma : \mathbf{p} \in P \mapsto \sigma(\mathbf{p}) \in U$ is called the *stereographic projection* with centre $\mathbf{z}$ on the unit sphere (see fig. 1). Its inverse map $\sigma^{-1} : \mathbf{u} \in U \mapsto \sigma^{-1}(\mathbf{u}) \in P$ is given by

$$\sigma^{-1}(\mathbf{u}) = (\quad u_0 - u_3 \quad u_1 \quad u_2 \quad 0 \quad)^\top. \tag{7}$$

The stereographic projection σ and its inverse map preserve circles [12]: The image of a circle or line on P under σ is a circle on U.

The image of a bilinear rational Bézier patch on the plane P under σ is a biquadratic Bézier patch on the unit sphere U. But stereographic projection does not yield all biquadratic patches on U as images of bilinear Bézier patches. (A counterexample can be found in Fink'92 [7].)

The *generalized stereographic projection* $\delta : \mathbf{e} \in \bar{E}^3 \mapsto \delta(\mathbf{e}) \in U$ where

$$\delta(\mathbf{e}) = \begin{pmatrix} e_0^2 + e_1^2 + e_2^2 + e_3^2 \\ 2e_0e_1 - 2e_2e_3 \\ 2e_1e_3 + 2e_0e_2 \\ e_1^2 + e_2^2 - e_0^2 - e_3^2 \end{pmatrix} \tag{8}$$

avoids this disadvantage [9]: *Any irreducible rational Bézier patch of degree $(2m, 2n)$ on the unit sphere U can be obtained as image of a patch of degree (m, n) in $\bar{E}^3$ under δ.*

The *hyperbolic projection* $\vartheta : \mathbf{e} \in \bar{E}^3 \mapsto \vartheta(\mathbf{e}) \in P$ where

$$\vartheta(\mathbf{e}) = \begin{pmatrix} e_0^2 + e_3^2 \\ e_0e_1 - e_2e_3 \\ e_1e_3 + e_0e_2 \\ 0 \end{pmatrix} \tag{9}$$

Figure 1: The generalized stereographic projection δ on the unit sphere

has been introduced in Dietz et al.'93 [9] in order to discuss the properties of the generalized stereographic projection. Some of these properties are:

(U1) The generalized stereographic projection δ is the composition of the hyperbolic projection ϑ with the stereographic projection σ: $\delta = \sigma \circ \vartheta$ (see fig. 1).

(U2) The set of all inverse images of a point $\mathbf{p} \in P$ under ϑ (and so of a point $\mathbf{u} \in U$ with $\mathbf{p} = \sigma^{-1}(\mathbf{u})$ under δ) forms the line

$$\lambda \mathbf{q} + \mu \mathbf{q}^{\perp} \quad (\lambda, \mu \in \mathbb{R}) \tag{10}$$

where $\mathbf{q}$ is an arbitrary preimage of $\mathbf{p}$ under ϑ, for example $\mathbf{q} = \mathbf{p}$.
The lines $\lambda \mathbf{q} + \mu \mathbf{q}^{\perp}$ $(\lambda, \mu \in \mathbb{R})$ are called the *projecting lines* of the hyperbolic projection. Each one of them passes through its image under ϑ and is perpendicular to the line connecting its image under ϑ and the origin. An arbitrary rotation around the z-axis maps projecting lines to projecting lines. Figure 2 shows the projecting lines of the hyperbolic projection.

(U3) The inverse image of a circle on P under ϑ (and so of a circle not passing through the centre $\mathbf{z}$ on U under δ) is a one-sheet-hyperboloid. The inverse image of a line on P under ϑ (and so of a circle through $\mathbf{z}$ on U under δ) is a hyperbolic paraboloid.
These one-sheet-hyperboloids and hyperbolic paraboloids carry two systems of lines (generators). The first system consists of projecting lines (10) of ϑ. The lines of the second system will be called the *conjugated lines* with respect to the first system. The image of a conjugated line under ϑ is the given circle on P.

Figure 2: The projecting lines of the hyperbolic projection

(U4) The image of a given non-projecting line under ϑ is a circle or a line on P. Thus the image of a given non-projecting line under δ is a circle on U.

(U5) Any plane E in $\bar{E}^3$ contains exactly one projecting line L_E. Consider two distinct non-projecting lines L_1, L_2 on the plane E. Their images under ϑ resp. δ intersect in the two points $\vartheta(L_1 \cap L_2)$ and $\vartheta(L_E)$ resp. in $\delta(L_1 \cap L_2)$ and $\delta(L_E)$. If L_1 and L_2 intersect on L_E, then the tangents of their images under ϑ resp. δ at $\vartheta(L_E)$ resp. $\delta(L_E)$ coincide.

The hyperbolic projection is a special *net projection.* Its projecting lines form an *elliptic linear congruence of lines* (see fig. 2). Linear congruences of lines and net projections have been studied in advanced geometry [13] [14].

2.2 The construction of Bézier patches from given boundaries

This section applies the discussed projections to the construction of spherical surface patches. The case of triangular surface patches has been outlined in [9], here we will focus on the tensor-product case.

The biquadratic Bézier patch. Every rational biquadratic Bézier patch lying on the unit sphere U can be generated with help of the generalized stereographic projection δ (see (8)). A rational bilinear patch can be found yielding the desired biquadratic patch when it is mapped

onto the unit sphere by δ.
Because the boundaries of a biquadratic patch are plane curves, they are circle segments when lying on U. The four boundary circles intersect in the patch corners — which we will call $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{p}_4$ — and in four additional points $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ and $\mathbf{q}_4$. The point $\mathbf{q}_i$ stands for the second intersection point of the two circles passing through $\mathbf{p}_i$.
Suppose, the four patch corners and boundary circle segments are given. The question is, whether there is always a biquadratic Bézier patch interpolating the desired boundaries and — if so — how its homogeneous control points can be constructed.
In Dietz et al.'93 [9] it has been shown that the boundary circles of a biquadratic patch on U cannot lie arbitrarily, but always fulfill the following condition:

- The four points $\mathbf{p}_1, \mathbf{p}_3, \mathbf{q}_2$ and $\mathbf{q}_4$ (or equivalently $\mathbf{p}_2, \mathbf{p}_4, \mathbf{q}_1$ and $\mathbf{q}_3$) are coplanar.

Thus, we have to ensure that the given boundary circles satisfy the stated condition, i.e.

$$\det(\mathbf{p}_1, \mathbf{p}_3, \mathbf{q}_2, \mathbf{q}_4) = 0, \tag{11}$$

otherwise the desired patch cannot be constructed and a patch of degree $(2, 4)$ has to be used.
We may define the boundaries by prescribing three points for each one, e.g. $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{u}_1$ for the first boundary and so on for the others. Then the second intersection points $\mathbf{q}_i$ have to be constructed first.
For regularity of the considered patch we will assume the points $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ and $\mathbf{q}_4$ not to lie on the patch boundaries. Now, we have to find a rational bilinear Bézier patch

$$\mathbf{y}(u,v) = B_0^1(u)\, B_0^1(v)\, \mathbf{c}_1 + B_1^1(u)\, B_0^1(v)\, \mathbf{c}_2 + B_0^1(u)\, B_1^1(v)\, \mathbf{c}_3 + B_1^1(u)\, B_1^1(v)\, \mathbf{c}_4$$

which is mapped to a biquadratic patch $\mathbf{x}(u, v)$ with the given corners and boundaries.
Consider the preimages under δ of the specified boundary circles on U. They are four doubly ruled surfaces, namely one-sheet hyperboloids or hyperbolic paraboloids (cf. (U3)), which intersect in the four projecting lines $\delta^{-1}(\mathbf{p}_1), \ldots, \delta^{-1}(\mathbf{p}_4)$ (see (10)). The points $\mathbf{c}_1, \ldots, \mathbf{c}_4$ have to lie on these lines, and the connecting lines of two neighboured control points must be conjugated generators of the corresponding doubly ruled quadric (U3).
One of the corner points, e.g. $\mathbf{c}_1$, can be chosen arbitrarily on the projecting line $\delta^{-1}(\mathbf{p}_1)$. Then the point $\mathbf{c}_2$ has to be determined so that the line segment $B_0^1(u)\, \mathbf{c}_1 + B_1^1(u)\, \mathbf{c}_2$ ($u \in [0, 1]$) is part of the conjugated generating line through $\mathbf{c}_1$. Because the boundary circles of the spherical patch should not pass through the second intersection points, that line segment additionally must not pass through the line $\delta^{-1}(\mathbf{q}_2)$. Analogously, the points $\mathbf{c}_3$ and $\mathbf{c}_4$ can be determined with help of the given second and third boundary circle.
Since condition (11) holds, the line segment between $\mathbf{c}_4$ and $\mathbf{c}_1$ is automatically part of a conjugated generator of the fourth preimage quadric. It may pass through the projecting

Figure 3: The construction of the control point $\mathbf{c}_{i+1}$

line $\delta^{-1}(\mathbf{q}_4)$, so that the fourth boundary circle passes through $\mathbf{q}_4$, or may not. This can be examined in the individual case. There does not exist a Bézier patch with the same first three boundary circle segments but complementary fourth boundary to the patch just constructed! A sufficient condition for the existence of a Bézier patch with four specified circle segments can be given as follows: the segment from $\mathbf{p}_1$ to $\mathbf{p}_3$ of the circle through $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{q}_2$ and $\mathbf{q}_4$ has to pass through either both or none of $\mathbf{q}_2$ and $\mathbf{q}_4$.
The construction of the biquadratic Bézier patch is summarized by the following algorithm:

Given: Corner points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and second intersection points $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$ on the sphere U.

- Determine preimages under δ of the given points (cf. U2), i.e.
  $\mathbf{r}_i := \sigma^{-1}(\mathbf{p}_i) \in \delta^{-1}(\mathbf{p}_i), \quad \mathbf{s}_i := \sigma^{-1}(\mathbf{q}_i) \quad$ (see (7)).
- Choose $\mathbf{c}_1 := \mathbf{r}_1$.
- Calculate the planes $\hat{\mathbf{v}}_i$ containing $\mathbf{c}_i$ and the projecting line $\delta^{-1}(\mathbf{q}_j)$, where $\mathbf{q}_j$ is a second intersection point belonging to the current boundary curve ($(i,j) \in \{(1,2),$ $(2,2),\ (3,4)\}$). The line $\delta^{-1}(\mathbf{q}_j)$ is the connecting line of $\mathbf{s}_j$ and $\mathbf{s}_j^{\perp}$ (see (2), (10)). Then intersect $\hat{\mathbf{v}}_i$ with the preimage line $\delta^{-1}(\mathbf{p}_{i+1})$ of the endpoint of the boundary circle segment to obtain the control point $\mathbf{c}_{i+1}$, see Figure 3. The line $\delta^{-1}(\mathbf{p}_{i+1})$ is the intersection of the planes $\hat{\mathbf{r}}_{i+1}$ and $\hat{\mathbf{r}}_{i+1}^{\perp}$.

$$\begin{array}{lll} \hat{\mathbf{v}}_1 = \mathbf{c}_1 \vee \mathbf{s}_2 \vee \mathbf{s}_2^{\perp} & \hat{\mathbf{v}}_2 = \mathbf{c}_2 \vee \mathbf{s}_2 \vee \mathbf{s}_2^{\perp} & \hat{\mathbf{v}}_3 = \mathbf{c}_3 \vee \mathbf{s}_4 \vee \mathbf{s}_4^{\perp} \\ \mathbf{c}_2 = \hat{\mathbf{v}}_1 \wedge \hat{\mathbf{r}}_2 \wedge \hat{\mathbf{r}}_2^{\perp} & \mathbf{c}_3 = \hat{\mathbf{v}}_2 \wedge \hat{\mathbf{r}}_3 \wedge \hat{\mathbf{r}}_3^{\perp} & \mathbf{c}_4 = \hat{\mathbf{v}}_3 \wedge \hat{\mathbf{r}}_4 \wedge \hat{\mathbf{r}}_4^{\perp} \end{array} \quad (12)$$

The line passing through $\mathbf{c}_i$ and $\mathbf{c}_{i+1}$ $(i = 1, 2, 3)$ is a conjugated generating line of the doubly ruled surface which is mapped to the specified boundary circle by δ.

Figure 4: The general biquadratic patch on the unit sphere

- Adjust the signs of the control points so that the boundaries of the spherical patch do not pass through $\mathbf{q}_1, \ldots, \mathbf{q}_4$, and scale them to a length of 1 so that the weights of the corner Bézier points of the patch on U are equal to 1.
- Map the rational bilinear patch $\mathbf{y}(u, v)$ with the Bézier points $\mathbf{c}_1, \ldots, \mathbf{c}_4$ onto U by the generalized stereographic projection δ to obtain the biquadratic patch with specified boundaries.

The expressions (12) can be computed using equation (4) or by [9]

$$\hat{\mathbf{v}}_1 = \langle \mathbf{c}_1, \hat{\mathbf{s}}_2 \rangle \hat{\mathbf{s}}_2^{\perp} - \langle \mathbf{c}_1, \hat{\mathbf{s}}_2^{\perp} \rangle \hat{\mathbf{s}}_2, \quad \mathbf{c}_2 = \langle \hat{\mathbf{v}}_1, \mathbf{r}_2 \rangle \mathbf{r}_2^{\perp} - \langle \hat{\mathbf{v}}_1, \mathbf{r}_2^{\perp} \rangle \mathbf{r}_2, \quad \text{and so on.} \tag{13}$$

The latter formulae (13) cause the line segment $\mathbf{c}_1 B_0^1(u) + \mathbf{c}_2 B_1^1(u)$ $(u \in [0,1])$ not to pass through $\delta^{-1}(\mathbf{q}_2)$, so that we do not need to care about the signs of the control points.
When the bilinear Bézier patch $\mathbf{y}(u, v)$ is mapped onto U, the control points of the resulting patch are obtained. In these computations products of Bernstein polynomials occur, which can be computed with help of the product formula

$$B_i^m(t)\, B_j^n(t) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}} B_{i+j}^{m+n}(t)\ . \tag{14}$$

Figure 4 shows a biquadratic spherical patch together with the circle on which the points $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{q}_2$ and $\mathbf{q}_4$ lie. This patch is neither a rotational patch (i. e. generated by rotating a circle) nor do the four boundaries intersect in one point of U — as they would do, if it was constructed by mapping a bilinear surface onto U with help of the “ordinary” stereographic projection. The biquadratic patch can be decomposed into two triangular patches of degree two [9]. But it is impossible to describe it by one quadratic parameterization, because otherwise its boundaries would intersect in a point of the sphere.

The patch of degree (2,4). We saw, that for a biquadratic Bézier patch on the unit sphere the four boundary circles cannot lie in arbitrarily given planes. Nevertheless, we can prescribe the fourth boundary too by using a rational patch of degree (2,4). Hence, we have to construct a patch

$$\mathbf{y}(u,v) = \sum_{i=0}^{1}\sum_{j=0}^{2} B_i^1(u)\, B_j^2(v)\, \mathbf{c}_{ij},$$

of degree (1,2) which is mapped by the generalized stereographic projection δ to the desired surface patch. For simplicity we choose three of the boundaries in exactly the same way as in the case of the biquadratic patch:

$$\mathbf{c}_{00} = \mathbf{c}_1, \quad \mathbf{c}_{10} = \mathbf{c}_2, \quad \mathbf{c}_{11} = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2), \quad \mathbf{c}_{12} = \mathbf{c}_3 \quad \text{and} \quad \mathbf{c}_{02} = \mathbf{c}_4$$

The fourth boundary is a rational quadratic Bézier curve, thus a conic section, defined by the control points $\mathbf{c}_{00}$, $\mathbf{c}_{01}$ and $\mathbf{c}_{02}$. The unknown point $\mathbf{c}_{01}$ has to be chosen so that the mapped curve is the circle through $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{q}_4$.
The preimage under δ of this circle is a doubly ruled quadric surface (U3) on which the points $\mathbf{c}_{00}$ and $\mathbf{c}_{02}$ are situated. In general, they do not lie on the same conjugated generating line of this quadric. But if an arbitrary plane containing $\mathbf{c}_{00}$ and $\mathbf{c}_{02}$ is intersected with the quadric surface, a conic section is obtained which is mapped by δ to the desired fourth boundary circle on U. Of course, the segment of this conic section between $\mathbf{c}_{00}$ and $\mathbf{c}_{02}$ can be described as a quadratic Bézier curve. This yields the middle control point $\mathbf{c}_{01}$.
First, we represent the above quadric as a bilinear rational Bézier surface. The point $\mathbf{d}_1$ (resp. $\mathbf{d}_4$) may stand for the intersection of the projecting line through $\mathbf{c}_{00}$ (resp. $\mathbf{c}_{02}$) and the conjugated generator through $\mathbf{c}_{02}$ (resp. $\mathbf{c}_{00}$). The conjugated lines connecting $\mathbf{c}_{00}$ and $\mathbf{d}_4$ resp. $\mathbf{d}_1$ and $\mathbf{c}_{02}$ pass through $\delta^{-1}(\mathbf{q}_4)$. With appropriate weights it holds that

$$\begin{aligned} B_0^1(u_0)\,\mathbf{c}_{00} + B_1^1(u_0)\,\mathbf{d}_4 &\in \delta^{-1}(\mathbf{q}_4) \quad \text{and} \\ B_0^1(u_0)\,\mathbf{d}_1 + B_1^1(u_0)\,\mathbf{c}_{02} &\in \delta^{-1}(\mathbf{q}_4) \qquad u_0 \notin [0,1]. \end{aligned} \tag{15}$$

The whole doubly ruled quadric can be now represented by the Bézier surface

$$\mathbf{r}(u,v) = B_0^1(u)\, B_0^1(v)\, \mathbf{c}_{00} + B_0^1(u)\, B_1^1(v)\, \mathbf{d}_1 + B_1^1(u)\, B_0^1(v)\, \mathbf{d}_4 + B_1^1(u)\, B_1^1(v)\, \mathbf{c}_{02}$$

$(u, v \in \mathbb{R} \cup \{\infty\})$. Hence, the quadratic curve

$$\mathbf{r}(t,t) = B_0^2(t)\,\mathbf{c}_{00} + \frac{1}{2} B_1^2(t)\,(\mathbf{d}_1 + \mathbf{d}_4) + B_2^2(t)\,\mathbf{c}_{02}, \qquad t \in [0,1]\,,$$

runs from $\mathbf{c}_{00}$ to $\mathbf{c}_{02}$ and lies completely on the preimage of the fourth given circle on U, see Figure 5. The missing Bézier control point is therefore $\mathbf{c}_{01} = \frac{1}{2}(\mathbf{d}_1 + \mathbf{d}_4)$.

Figure 5: The construction of the fourth boundary

Thus the Bézier patch $\mathbf{y}(u,v)$ of degree (1,2) which is mapped to the given spherical patch with arbitrary plane boundaries has been determined. Figure 6 illustrates the situation for a surface patch whose boundary curves ought to lie in planes parallel to a coordinate axis. The biquadratic patch obtained by prescribing three boundary circles is a patch of revolution (Fig.6a). The corresponding patch of degree (2,4) (Fig.6b) realizes the fourth boundary too. Either pair of opposite boundaries can be chosen to have degree four, the remaining two are quadratic.

2.3 Interpolation on the sphere

Now, we consider the problem of finding a rational spherical Bézier curve interpolating a set of given points on the unit sphere U. Suppose, $2n+1$ points $\mathbf{p}_0, \ldots, \mathbf{p}_{2n}$ on U with parameters $t_0 < t_1 < \ldots < t_{2n}$ are given. We are searching for a Bézier curve

$$\mathbf{x}(t) = \sum_{i=0}^{2n} B_i^{2n}(t)\, \mathbf{b}_i \tag{16}$$

satisfying $\mathbf{x}(t_i) \sim \mathbf{p}_i$ for $i = 0, \ldots, 2n$. ($\sim$ stands for linear dependence of the homogeneous coordinate vectors.) It has been shown that this problem leads to a system of *linear* equations and that the interpolating curve is uniquely determined [9].
Because the generalized stereographic projection δ yields every curve of degree $2n$ on U as image of a degree n curve, we have to find a rational curve

$$\mathbf{y}(t) = \sum_{i=0}^{n} B_i^{n}(t)\, \mathbf{c}_i \tag{17}$$

(a) (b)

Figure 6: Patches of degree (2,2) (a) resp. (2,4) (b) with prescribed boundaries

of degree n passing through the preimages $\delta^{-1}(\mathbf{p}_i)$ of the given points. Every projecting line $\delta^{-1}(\mathbf{p}_i)$ of δ is the intersection of two planes $\hat{\mathbf{r}}_i$ and $\hat{\mathbf{r}}_i^{\perp}$ where $\mathbf{r}_i \in \delta^{-1}(\mathbf{p}_i)$ can be chosen arbitrarily (cf. (2) and (10)). Thus, the curve $\mathbf{y}(t)$ must lie in both planes for the parameter value t_i:

$$\langle \hat{\mathbf{r}}_i, \mathbf{y}(t_i) \rangle = 0 \qquad \langle \hat{\mathbf{r}}_i^{\perp}, \mathbf{y}(t_i) \rangle = 0 \quad , \quad i = 0, \ldots, 2n. \tag{18}$$

These are $4n+2$ linear homogeneous equations with $4n+4$ unknowns, namely four components of each Bézier point $\mathbf{c}_0, \ldots, c_n$. There are two linearly independent solutions satisfying the system of equations, but they both are mapped to the same spherical curve $\mathbf{x}(t)$ when applying the generalized stereographic projection δ.

In certain cases, $\mathbf{y}(t_i) = \mathbf{0}$ may occur, which means the curve $\mathbf{y}(t)$ has a *base point* at parameter value t_i. (The homogeneous curve $\mathbf{y}(t)$ passes for $t = t_i$ through the origin $\mathbf{0} \in \mathbb{R}^4$.) Then the curve $\mathbf{x}(t) = \delta(\mathbf{y}(t))$ generally does not interpolate $\mathbf{p}_i$, because this point is *inaccessible* in the context of the interpolation problem.

When the control points of $\mathbf{y}(t)$ have been computed, the control points of $\mathbf{x}(t)$ follow directly out of (8) and the multiplication formulas (14) for Bernstein polynomials. A Bézier curve of degree 6 interpolating 7 given points is depicted in Figure 7.

3 Rational patches on the hyperbolic paraboloid

This section discusses rational patches on the hyperbolic paraboloid $\mathbf{x}^\top H \mathbf{x} = 0$. The letter H means the set of all points of this quadric surface. The set of all points of the plane $r_0 = r_3$

Figure 7: Interpolating Bézier curve with control polygon

(i.e. $z = 1$) is denoted by R.

3.1 The generalized stereographic projection

Analogously to the discussion of the unit sphere, this section starts with a generalization of the stereographic projection.

Let $\mathbf{c}$ denote the point $(\,0\,0\,0\,1\,)^\top$, i.e. the infinite point of the z-axis. The line connecting an arbitrary point $\mathbf{r} \in R$ with $\mathbf{c}$ intersects the hyperbolic paraboloid H in exactly two points: in $\mathbf{c}$ and a second one $\tau(\mathbf{r})$ where

$$\tau(\mathbf{r}) = \begin{pmatrix} r_0^2 \\ r_0 r_1 \\ r_0 r_2 \\ r_1 r_2 \end{pmatrix}. \tag{19}$$

The map $\tau : \mathbf{r} \in R \mapsto \tau(\mathbf{r}) \in H$ is called the *stereographic projection* with centre $\mathbf{c}$ on the hyperbolic paraboloid (see fig. 8). Its inverse map $\tau^{-1} : \mathbf{h} \in H \mapsto \tau^{-1}(\mathbf{h}) \in R$ is given by

$$\tau^{-1}(\mathbf{h}) = (\begin{array}{cccc} h_0 & h_1 & h_2 & h_0 \end{array})^\top. \tag{20}$$

The intersection of an arbitrary plane with the hyperbolic paraboloid H is a hyperbola (or a parabola or a pair of lines). The two asymptotes of this hyperbola are parallel to the xz- resp. yz-plane of the coordinate system. The stereographic projection τ and its inverse map preserve this parallelism: If the two asymptotes of a hyperbola on R are parallel to the x- resp. y-axis, then the image of the hyperbola under τ is a hyperbola on H.

Again, the image of a bilinear rational Bézier patch on the plane R under τ is a biquadratic Bézier patch on the hyperbolic paraboloid H in general. But stereographic projection does not yield all biquadratic patches on H as images of bilinear Bézier patches. The *generalized stereographic projection* $\psi : \mathbf{e} \in \bar{E}^3 \mapsto \psi(\mathbf{e}) \in H$ where

$$\psi(\mathbf{e}) = \begin{pmatrix} e_0 e_3 \\ e_1 e_3 \\ e_0 e_2 \\ e_1 e_2 \end{pmatrix} \tag{21}$$

avoids this disadvantage [9]: *Any irreducible rational Bézier patch of degree $(2m, 2n)$ on the hyperbolic paraboloid can be obtained as image of a rational Bézier patch in $\bar{E}^3$ under ψ. The coordinates e_0 and e_1 resp. e_2 and e_3 of this patch have the degrees (m_1, n_1) resp. (m_2, n_2) satisfying $m_1 + m_2 = 2m$ and $n_1 + n_2 = 2n$.*
Analogous to the previous section, the *axial projection* $\alpha : \mathbf{e} \in \bar{E}^3 \mapsto \alpha(\mathbf{e}) \in R$ where

$$\alpha(\mathbf{e}) = \begin{pmatrix} e_0 e_3 \\ e_1 e_3 \\ e_0 e_2 \\ e_0 e_3 \end{pmatrix} \tag{22}$$

is introduced in order to discuss the properties of the generalized stereographic projection. Some of these properties are:

(H1) The generalized stereographic projection ψ is the composition of the axial projection α with the stereographic projection τ: $\psi = \tau \circ \alpha$ (see fig. 8).

(H2) The set of all inverse images of a point $\mathbf{r} \in R$ under α (and so of a point $\mathbf{h} \in H$ with $\mathbf{r} = \tau^{-1}(\mathbf{h})$ under ψ) forms the line

$$\lambda \mathbf{q} + \mu \mathbf{q}^* \quad (\lambda, \mu \in \mathbb{R}) \tag{23}$$

where $\mathbf{q}$ is an arbitrary preimage of $\mathbf{r}$ under α, for example $\mathbf{q} = \mathbf{r}$.
The lines $\lambda \mathbf{q} + \mu \mathbf{q}^*$ $(\lambda, \mu \in \mathbb{R})$ are called the *projecting lines* of the axial projection. Each one of them passes through its image under α and through the x-axis. Additionally, it is parallel to the yz-plane. Figure 9 shows the projecting lines of the axial projection.

(H3) Consider the hyperbolas on R whose asymptotes are parallel to the x- resp. y-axis. (Its images under τ are the hyperbolas on H.) The inverse images of these hyperbolas under α are one-sheet-hyperboloids. Additionally, the inverse images of the lines on R under α (and so of the parabolas or lines on H under ψ) are hyperbolic paraboloids or planes. These one-sheet-hyperboloids and hyperbolic paraboloids carry two systems of lines

Figure 8: The generalized stereographic projection ψ on the hyperbolic paraboloid

Figure 9: The projecting lines of the axial projection

(generators). The first system consists of projecting lines (23) of α. The lines of the second system will be called the *conjugated lines* with respect to the first system. The image of a conjugated line under α is a hyperbola or a line on R.

(H4) The image of a given non-projecting line under α is a hyperbola, whose asymptotes are parallel to the x- resp. y-axis of the coordinate system, or a line on R. Thus the image of a given non-projecting line under ψ is a hyperbola, a parabola or a line on H.

(H5) Any plane E in $\bar{E}^3$ contains at least one projecting line L_E. If the plane E does not contain the x-axis and if it is not parallel to the yz-plane, then the line L_E is unique. The second part of (U5) holds similarly.

Analogous to section 2.1, the axial projection is a special *net projection*. Its projecting lines (see fig. 9) form a *hyperbolic linear congruence of lines* [13] [14].
The next section applies the discussed projections:

3.2 The construction of Bézier patches from given boundaries

Rational and integral tensor product surface patches on H with plane boundary curves shall be constructed now. Unlike the spherical case, there are also patches of degree $(1,1)$ and degree $(1,2)$ — either rational or integral — lying completely on H.
The hyperbolic paraboloid is a doubly ruled quadric and carries two systems of generators. Such a straight line on H is obtained by the generalized stereographic projection ψ (see (21)) as image of a line (curve) which is parallel to the yz-plane or which lies in a plane containing the x-axis. We get integral representations when the curve or surface to be mapped onto H is parallel to the xy-plane.

The bilinear patch arises as image of another bilinear patch

$$\mathbf{y}(u,v) = B_0^1(u)\,B_0^1(v)\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix} + B_1^1(u)\,B_0^1(v)\begin{pmatrix}b_0\\b_1\\a_2\\a_3\end{pmatrix} + B_0^1(u)\,B_1^1(v)\begin{pmatrix}a_0\\a_1\\b_2\\b_3\end{pmatrix} + B_1^1(u)\,B_1^1(v)\begin{pmatrix}b_0\\b_1\\b_2\\b_3\end{pmatrix} \tag{24}$$

whose boundaries lie in planes parallel to the yz-plane resp. in planes containing the x-axis. When ψ is applied, we get a Bézier patch with formal degree $(2,2)$, but it is a degree-elevated representation of a bilinear patch. The patch is integral iff $a_0 = b_0$ and $a_3 = b_3$ holds. The points $\mathbf{a} = (a_0, a_1, a_2, a_3)^\top$ and $\mathbf{b} = (b_0, b_1, b_2, b_3)^\top$ can be chosen as arbitrary preimages under ψ of given opposite corner points on H (see (H2)).
We can derive from representation (24) how (in case of existence) a linear curve on H connecting two points $\mathbf{p}_1$, $\mathbf{p}_2$ can be constructed: Choose two arbitrary preimage points $\mathbf{r}_1$, $\mathbf{r}_2$ on

Figure 10: Tensor product patch of degree (1,2) on the hyperbolic paraboloid

the projecting lines $\psi^{-1}(\mathbf{p}_1)$ and $\psi^{-1}(\mathbf{p}_2)$, adjust the weight of $\mathbf{r}_2$ so that the first resp. last two components are equal to those of $\mathbf{r}_1$, then $B_0^1(t)\,\mathbf{r}_1 + B_1^1(t)\,\mathbf{r}_2$ yields the desired linearly parameterized line on H.

Patches of degree (1,2). Suppose, four boundary curves on H are given. Two opposite curves are straight lines, the other two being parabolas or hyperbolas connecting the straight lines. Then we can always find a rational Bézier patch of degree $(1,2)$ whose boundaries are segments of the above curves.
A quadratic boundary curve (e.g. between the corner points $\mathbf{p}_1$ and $\mathbf{p}_2$) can be constructed analogously to the spherical case as image under ψ of a linear curve: the preimage of the specified boundary is generally a doubly ruled surface whose conjugated generating lines are mapped by ψ to the given boundary curve (H3). Hence, two points on the projecting lines $\psi^{-1}(\mathbf{p}_1)$ and $\psi^{-1}(\mathbf{p}_2)$ can be chosen, so that their connecting line is a conjugated generator of the preimage surface. The weights of the points are arbitrary.
Due to the fact that for a linear curve the weight of the endpoint is uniquely determined but the endpoint itself can be chosen (on the corresponding projecting line), this allows a construction of a bilinear patch which is mapped to a patch of degree $(1,2)$ enclosed by four specified boundary curves. A patch that has been constructed this way is shown in Figure 10.

Biquadratic patches. The construction of biquadratic Bézier patches on H is analogously to that of patches on U. Generally, a patch with four given plane boundary curves does exist only, if the patch corners $\mathbf{p}_1$, $\mathbf{p}_3$ and the second intersection points $\mathbf{q}_2$ and $\mathbf{q}_4$ are coplanar, i.e. they lie on a conic section or on two intersecting straight lines on H. An exception is

Figure 11: The general biquadratic patch on the hyperbolic paraboloid

made by those patches one of whose boundary curves is a line segment. The constructions of linear and quadratic boundary curves on H directly show, that in this case the location of the boundaries is arbitrary.
Figure 11 illustrates the biquadratic patch on H with the points $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{q}_2$ and $\mathbf{q}_4$ lying in a plane, in this case on a hyperbola.
If the bilinear patch which is mapped on H by ψ is parallel to the xy-plane, an integral Bézier patch is obtained having four boundary parabolas resp. line segments.

Patches of degree (2,4). If four arbitrary boundary hyperbolas or parabolas are prescribed, a patch of degree $(2,4)$ has to be used. The preimage patch of degree $(1,2)$ yielding the desired surface patch is constructed analogously to the spherical case.

3.3 Interpolation on the hyperbolic paraboloid

Now, we construct a rational curve on H interpolating given points. The curve of degree $2n$ can be set up as image under ψ of a degree n curve $\mathbf{y}(t)$ which passes through the $2n+1$ preimage lines $\psi^{-1}(\mathbf{p}_i)$ $(i=0,\ldots,2n)$ of the given points $\mathbf{p}_0,\ldots,\mathbf{p}_{2n}$ at the given parameter values $t_0 < t_1 < \ldots < t_{2n}$. The line $\psi^{-1}(\mathbf{p}_i)$ is the intersection of the planes $\hat{\mathbf{r}}_i^*$ and $\hat{\mathbf{r}}_i^{**}$ (see (2) and (23)), where $\mathbf{r}_i \in \psi^{-1}(\mathbf{p}_i)$. Thus, the interpolation problem for the curve $\mathbf{y}(t)$ is a linear problem and is given by

$$\langle \hat{\mathbf{r}}_i^*, \mathbf{y}(t_i)\rangle = 0 \qquad \langle \hat{\mathbf{r}}_i^{**}, \mathbf{y}(t_i)\rangle = 0 \quad , \quad i=0,\ldots,2n. \tag{25}$$

Figure 12: Interpolating curve on the hyperbolic paraboloid

These $4n+2$ equations can be decomposed into two systems of linear equations, namely

$$\langle \hat{\mathbf{r}}_i, \mathbf{y}^*(t_i)\rangle = 0 \quad , \quad i=0,\ldots,2n \qquad\qquad \langle \hat{\mathbf{r}}_i, \mathbf{y}^{**}(t_i)\rangle = 0 \quad , \quad i=0,\ldots,2n \tag{26}$$

consisting of $2n+1$ equations with $2n+2$ unknowns (the components of the control points of $\mathbf{y}(t)$) each. The first equation system yields the 0th- and 1st-components of the curve $\mathbf{y}(t)$, the second system yields the 2nd- and 3rd-components. If $\mathbf{y}^*(t_i) = 0$ or $\mathbf{y}^{**}(t_i) = 0$ holds, the curve $\mathbf{x}(t) = \psi(\mathbf{y}(t))$ has a *base point* for $t = t_i$, i.e. $\mathbf{x}(t_i) = 0$. Then the point $\mathbf{p}_i$ is *inaccessible* in this interpolation problem and is not interpolated by $\mathbf{x}(t)$.
Applying the generalized stereographic projection ψ to the curve $\mathbf{y}(t)$ yields the interpolating curve $\mathbf{x}(t)$. Its Bézier points can be easily computed. The constructed curve can have points at infinity and — moreover — points at infinity can be interpolated too.
Contrarily to the spherical case not every curve $\mathbf{x}(t)$ of degree $2n$ is obtained as image under ψ of a degree n curve $\mathbf{y}(t)$. The components $\mathbf{y}^*(t)$ and $\mathbf{y}^{**}(t)$ can be given different degrees which have to sum up to $2n$. But we see from (26), that this would cause one of the equation systems to gain additional degrees of freedom, the other in return could be unsolvable.
If we set up $\mathbf{y}(t)$ as curve of degree n we can show that every solution leads to the same interpolating curve $\mathbf{x}(t)$. Figure 12 shows a curve of degree four interpolating five given points.

4 B-spline curves and surfaces

We will now extend our constructions to B-spline curves and tensor-product B-spline surface patches on quadrics. We get the corresponding curve or surface representation if we replace for instance in (17) or (5) the Bernstein polynomials by B-spline basis functions $N_{i.k}(\tau)$ with k as order (degree $k-1$). The parameter τ may be defined over a knot sequence T. In the interior of T all knots may have multiplicity 1. Analogously to (14) we need product formulae for the B-spline functions $N_{i.k}(\tau)$. For such products we can set

$$N_{i.k}(\tau) \cdot N_{j.k}(\tau) = \sum_{m=1}^{M} \alpha_m^{(i.j)} \, N_{(s_{i.j}+m).(2k-1)}(\tau) \quad (i \leq j) \tag{27}$$

while multiplication of two functions of degree $k-1$ leads to a function of order $2k-1$. The coefficients $\alpha_m^{(i.j)}$ and the index shift $s_{i.j}$ have to be determined.
Because the B-spline functions $N_{i.k}$ have continuity of order $k-2$ with our assumption for the knot vector, the product of two basis functions must have the same continuity class. Therefore each knot in the knot sequence of $N_{m.2k-1}$ must have multiplicity k. The coefficients $\alpha_m^{(i.j)}$ can be determined recursively with help of wellknown recursiv definition of B-spline functions [15] [16]. The factors $\alpha_m^{(i.j)}$ vanish if $j \notin \{i, .., i+k-1\}$ otherwise M is determined by

$$\begin{array}{llll} \text{for} \quad j = i & : & M = k(k-3)+3 & \\ \text{for} \quad j = i+\beta & : & M = k(k-(\beta+1))+1 & \beta \in \{1, .., k-1\}. \end{array} \tag{28}$$

This formula has an asymmetry: for the product with $i = j$ two additional coefficients appear at the beginning and at the end of the sequence $\{i, .., i+k-1\}$.
Consider a closed B-spline curve of order $k = 3$ with a uniform knot sequence $(0, 1, 2, 3, \dots)$ in the parameter space $\bar{E}^3$. Applying the generalized stereographic projection (8) to this curve yields a closed B-spline curve of order 5 with the knot sequence $(0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \dots)$ on the sphere. Its homogeneous control points can be computed with help of the following product table for the basis functions:

$$\begin{array}{lclclclcl} (N_{i.3})^2 & = & \frac{1}{2} N_{3i+2.5} & + & \frac{3}{4} N_{3i+3.5} & + & \frac{1}{2} N_{3i+4.5}, & & \\ N_{i.3} \cdot N_{i+1.3} & = & \frac{1}{24} N_{3i+3.5} & + & \frac{1}{4} N_{3i+4.5} & + & \frac{1}{4} N_{3i+5.5} & + & \frac{1}{24} N_{3i+6.5}, \\ N_{i.3} \cdot N_{i+2.3} & = & \frac{1}{24} N_{3i+6.5}. & & & & & & \end{array}$$

The number M in (27) is reduced for *open* spline curves: then we have multiplicity k at the boundaries of the knot sequence and therefore M must be lower. For example, applying the generalized sterographic projection (8) to an open B-spline curve of order $k = 3$ with the knot sequence $(0, 0, 0, 1, 2, 3, 4, 5, 5, 5)$ in $\bar{E}^3$ yields an open B-spline curve of order 5 with the knot sequence $(0, 0, 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5)$ on the sphere. For the

B-spline basis functions we get the following products [15] [16]

$$
\begin{aligned}
(N_{0.3})^2 &= N_{0.5} \quad, \\
(N_{1.3})^2 &= \tfrac{2}{3}N_{2.5} + \tfrac{1}{2}N_{3.5} \quad, \\
(N_{2.3})^2 &= \tfrac{1}{2}N_{4.5} + \tfrac{3}{4}N_{5.5} + \tfrac{1}{2}N_{6.5} \quad, \\
(N_{3.3})^2 &= \tfrac{1}{2}N_{7.5} + \tfrac{3}{4}N_{8.5} + \tfrac{1}{2}N_{9.5} \quad, \\
(N_{4.3})^2 &= \tfrac{1}{2}N_{10.5} + \tfrac{2}{3}N_{11.5} \quad, \\
(N_{5.3})^2 &= N_{13.5} \quad,
\end{aligned}
$$

$$
\begin{aligned}
N_{0.3}\cdot N_{1.3} &= \tfrac{1}{2}N_{1.5} + \tfrac{1}{12}N_{2.5} \quad, \\
N_{1.3}\cdot N_{2.3} &= \tfrac{1}{4}N_{3.5} + \tfrac{1}{4}N_{4.5} + \tfrac{1}{24}N_{5.5} \quad, \\
N_{2.3}\cdot N_{3.3} &= \tfrac{1}{24}N_{5.5} + \tfrac{1}{4}N_{6.5} + \tfrac{1}{4}N_{7.5} + \tfrac{1}{24}N_{8.5} \quad, \\
N_{3.3}\cdot N_{4.3} &= \tfrac{1}{24}N_{8.5} + \tfrac{1}{4}N_{9.5} + \tfrac{1}{4}N_{10.5} \quad, \\
N_{4.3}\cdot N_{5.3} &= \tfrac{1}{12}N_{11.5} + \tfrac{1}{2}N_{12.5} \quad,
\end{aligned}
$$

$$
\begin{aligned}
N_{0.3}\cdot N_{2.3} &= \tfrac{1}{12}N_{2.5} \quad, \\
N_{1.3}\cdot N_{3.3} &= \tfrac{1}{24}N_{5.5} \quad, \\
N_{2.3}\cdot N_{4.3} &= \tfrac{1}{24}N_{8.5} \quad \text{and} \\
N_{3.3}\cdot N_{5.3} &= \tfrac{1}{12}N_{11.5} \quad .
\end{aligned}
$$

For modelling with B-splines on the sphere we consider the interpolation problem: Let $2n+1$ points $\mathbf{p}_i$ on the sphere with parameter values t_i $(i = 0, \ldots, 2n)$ be given. The lines $\delta^{-1}(\mathbf{p}_i)$ in the parameter space $\bar{E}^3$ have the equations (10), thus we get the same interpolation conditions (18) as in the Bernstein case: Let

$$\mathbf{y}(t) = \sum_{j=0}^{n} N_{j.k}(t)\,\mathbf{c}_j$$

be the B-spline curve which interpolates the lines $\delta^{-1}(\mathbf{p}_i)$ in $\bar{E}^3$ with the unknown homogeneous control points $\mathbf{c}_j$ and the B-spline basis functions $N_{j.k}(t)$ of order k (degree $k-1$). The knot sequence may have only values with multiplicity 1 in the interior of the parameter domain. With help of the pair of equations (18) we receive a $(4n+4)\times(4n+2)$ homogeneous linear system for the $(4n+4)$ unknown components of the control points $\mathbf{c}_j$. Thus two values (components) of the control points can be chosen completly free – nevertheless the image of $\mathbf{y}$ on the sphere is unique [9].

After determining the control points $\mathbf{c}_j$ with (18) we have to map the curve $\mathbf{y}(t) \in \bar{E}^3$ back to the sphere with help of the generalized stereographic projection (8). Now we have to

Figure 13: A closed spherical B-spline curve of order 5 with 3 segments interpolating 9 spherical points and the control polygon.

contemplate that there are different product formulae for the B-spline functions for open and for closed curves. Thus we get the following

Result 1. *If* $2n+1$ *points* $\mathbf{p}_0, \ldots \mathbf{p}_{2n}$ *on the sphere* U *with parameters* $t_0, \ldots, t_{2n} \in I\!R$ *(*$t_i \neq t_j$ *for* $i \neq j$*) are given, then there exists exactly one* closed *spherical rational B-spline curve of order* $2k-1$

$$\mathbf{x}(t) = \sum_{i=0}^{M} N_{i.2k-1}(t)\,\mathbf{d}_i$$

with $M = k(n+1) - 1$ *and the homogeneous control points* $\mathbf{d}_i$ *satisfying the homogeneous interpolation conditions (18)* (*). *(Note that again base points (cf. section 2.3) may occur !) Each knot of its knot vector has the multiplicity* k*. The knot vector of the preimage* $\mathbf{y}(t)$ *of* $\mathbf{x}(t)$ *under* δ *contains only parameter values of multiplicity* 1*.*

For the proof we consider the knot sequence: For the closed curve $\mathbf{y}(t)$ we have the sequence $(\tau_0, \ldots, \tau_n)$ with the multiplicity 1 for each value. The knots of the knot sequence of the image $\mathbf{x}(t)$ have the multiplicity k, that means we need for the same parametric interval $k(n+1)$ B-spline basis functions. Because we begin to summarize with 0 we have $M = k(n+1) - 1$.

Figure 13 contains an example for $k = 3, n = 4$ and $M = 14$ interpolating 9 given points on the sphere.
For open curves we have the following

*The parameter values t_i of the given points are assumed to be distributed as uniform as possible over the knot sequence. (Each interval of the knot sequence should contain "nearly the same number" of parameters t_i of given points.) Then the system of equations (18) has maximal rank.

Figure 14: An open spherical B-spline curve of order 5 with 4 segments interpolating 11 points and the control polygon.

Result 2. *If* $2n+1$ *points* $\mathbf{p}_0, \ldots, \mathbf{p}_{2n}$ *on the sphere* U *with parameters* $t_0, \ldots, t_{2n} \in I\!R$ $(t_i \neq t_j)$ *for* $i \neq j$ *are given, then there exists exactly one* open *spherical rational B-spline curve of order* $2k-1$

$$\mathbf{x}(t) = \sum_{i=0}^{M} N_{i.2k-1}(t)\, \mathbf{d}_i$$

with $M = k(n-k+3)-2$ *and the homogeneous control points* $\mathbf{d}_i$ *satisfying the homogeneous interpolation conditions (18) (*). Each interior knot of the knot sequence has the multiplicity* k*. The knot vector of the preimage* $\mathbf{y}(t)$ *of* $\mathbf{x}(t)$ *under* δ *consists of two boundary values with the multiplicity* k *and of interior values with the multiplicity* 1.

For the proof we have to remark, that an open curve $\mathbf{y}(t)$ has a knot sequence with $s = n-k+2$ segments, i.e. $s = n-k+1$ interior knot values. These values have the multiplicity k, thus we have $k(n-k+1)$ different B-spline basis functions in the interior. The two boundary values have the multiplicity $2k-1$, therefore we have totally $k(n-k+1)+2k-1$ different basis functions on the given parametric interval. Because we summarize from 0 we get

$$M = k(n-k+1) + 2k - 2 = k(n-k+3) - 2.$$

Figure 14 contains an example with $k = 3, n = 5$ and $M = 13$ interpolating 11 given points on the sphere.

Figure 15: A tensor-product B-spline surface of order 5 consisting of 2×2-segments. The boundary curves are given by points on the sphere.

If we extend these results to tensor-product surface patches we get for open patches the following

Result 3. *If* $(2n_u + 1) \cdot (2n_v + 1)$ *points* $\mathbf{p}_l$ *on the sphere* U *with parameters* $(u_l, v_l) \in I\!R^2$ *are given, then there exists exactly one quadrilateral spherical rational tensor-product B-spline patch of order* $(2k-1, 2k-1)$

$$\mathbf{x}(u,v) = \sum_{i=0}^{M_u} \sum_{j=0}^{M_v} N_{i.2k-1}(u)\, N_{j.2k-1}(v)\, \mathbf{d}_{i.j}$$

with $M_u = k(n_u - k + 3) - 2$, $M_v = k(n_v - k + 3) - 2$ *and the homogeneous control points* $\mathbf{d}_{i.j}$ *satisfying the homogeneous interpolation conditions* (**). *The interior knots of the knot sequences of* $\mathbf{x}(u,v)$ *have the multiplicity* k*, the interior knots of the knot sequences of the preimage* $\mathbf{y}(u,v)$ *under* δ *have the multiplicity* 1.

Another way to get a spherical tensor-product surface patch is to determine the boundary curves of the desired patch with help of interpolation of given points on the boundary curves with respect to result 1 or 2. Thus we have after solving the corresponding homogenous system the control points of the preimages $\mathbf{y}(t)$ of the required spherical curves. Now we can choose suitably the interior control points of the preimage under δ of the required spherical patch (figure 15). Note that the surface patch in figure 15 is only continuous of order 1 (C^1), but *geometric* continuous of order ∞ (G^∞) !

**The assumption (*) should hold similarly.

Figure 16: An open B-spline curve of order 5 with 2 segments interpolating 7 points on a hyperbolic paraboloid with control polygon.

We can extend the method of interpolation by Bezier curves to B-spline curves and surfaces on the hyperbolic paraboloid (or on the hyperboloid of revolution, see chapter 5). Figure 16 contains an open B-spline curve of order 5 with 2 segments interpolating 7 points on a hyperbolic paraboloid.

5 Extension to other quadrics

Consider an arbitrary nondegenerated quadric surface $\mathbf{x}^\top B\mathbf{x} = 0$ (where B is a symmetric nonsingular (4,4)-matrix) in $\bar{E}^3$. In this section, a projective map $\pi : \mathbf{x} \mapsto \pi(\mathbf{x}) = P\mathbf{x}$ (where P is a nonsingular (4,4)-matrix) mapping the given quadric surface to the unit sphere or to the hyperbolic paraboloid is constructed in order to extend the results of the preceding sections. The image of the given quadric under π is the unit sphere U or the hyperbolic paraboloid H iff $B = P^\top UP$ or $B = P^\top HP$ hold, respectively.

Applying principal axes transformation to matrix B yields an orthonormal matrix Q satisfying $B = Q^\top DQ$ where the diagonal matrix

$$D = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{29}$$

contains the eigenvalues $\lambda_i \in \mathrm{I\!R}$ $(i = 0, 1, 2, 3)$ of B. The given quadric surface $\mathbf{x}^\top B\mathbf{x} = 0$ was assumed to be nondegenerated. Thus all eigenvalues are not equal to zero.

First case. All eigenvalues have the same sign. In this case, the given quadric surface does not contain any real points.

Second case. Exactly one eigenvalue is positive. Let without loss of generality $\lambda_0 = (\xi_0)^2$, $\lambda_1 = -(\xi_1)^2$, $\lambda_2 = -(\xi_2)^2$ and $\lambda_3 = -(\xi_3)^2$ ($\xi_i \in \mathbb{R}$) be assumed. The required projective map π_U is given by $P_U = D_U Q$ where

$$D_U = \begin{pmatrix} \xi_0 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & \xi_3 \end{pmatrix}. \tag{30}$$

It satisfies $B = P_U^\top U P_U$, i.e. it maps the given quadric to the unit sphere U. The given quadric is an oval one.

Third case. Exactly two eigenvalues are positive. Let without loss of generality $\lambda_0 = (\zeta_0)^2$, $\lambda_1 = -(\zeta_1)^2$, $\lambda_2 = -(\zeta_2)^2$ and $\lambda_3 = (\zeta_3)^2$ ($\zeta_i \in \mathbb{R}$) be assumed. The required projective map π_H is given by $P_H = \frac{1}{\sqrt{2}} D_H Q$ where

$$D_H = \begin{pmatrix} \zeta_0 & \zeta_1 & 0 & 0 \\ 0 & 0 & \zeta_2 & \zeta_3 \\ 0 & 0 & \zeta_2 & -\zeta_3 \\ \zeta_0 & -\zeta_1 & 0 & 0 \end{pmatrix}. \tag{31}$$

It satisfies $B = P_H^\top H P_H$, i.e. it maps the given quadric to the hyperbolic paraboloid H. The given quadric is a doubly-ruled one.

Fourth case. Exactly three eigenvalues are positive. This case can be reduced to the second one by discussing the equation $\mathbf{x}^\top(-B)\mathbf{x} = 0$ instead of $\mathbf{x}^\top B\mathbf{x} = 0$. (Both equations describe the same quadric.)

With help of the constructed projective maps π_U resp. π_H, the given quadric surface $\mathbf{x}^\top B\mathbf{x} = 0$ (and the given boundaries of the required patches) can be mapped to the unit sphere U resp. to the hyperbolic paraboloid H. Now the methods derived in the previous sections can be applied in order to construct rational patches on U resp. H from given boundaries. Applying $\pi^{-1} : \mathbf{x} \mapsto \pi^{-1}(\mathbf{x}) = P^{-1}\mathbf{x}$ to these patches yields the required surface representations on the given quadric surface.

As example, figure 17 shows an interpolating rational B-spline curve of order 5 with 2 segments interpolating 7 points on the hyperboloid of revolution. First the given points have been mapped to the hyperbolic paraboloid with help of the projective map π_H (see (31)). Then the systems of equations (26) have been solved and an interpolating B-spline curve on H has been found. Applying the projective map π_H^{-1} has yielded the required curve on the hyperboloid of revolution.

Figure 17: B-spline curve of order 5 with 2 segments interpolating 7 points on the hyperboloid of revolution

Conclusion

Detailed constructions of rational patches on the unit sphere and on the hyperbolic paraboloid have been presented in this paper. These constructions are based on the use of generalized stereographic projections. The method has been applied to rational B-spline surface patches. The results have been extended to arbitrary nondegenerated quadric surfaces by constructing an appropriate projective transformation mapping a given oval or doubly ruled quadric surface to the unit sphere or to the hyperbolic paraboloid, respectively.
The method can be applied to arbitrary rational parametric representations of curves and surfaces. Figure 18 shows a five-sided patch on the sphere. First in the parameter space $\bar{E}^3$ a five-sided patch was constructed with a method introduced by Sabin [17] and then this patch was mapped with δ (see (8)) to the sphere.
Further research will complete the results by discussing degenerated quadrics like cones and cylinders. The generalized stereographic projection will be applied to the approximation of given point sets by rational curves on quadrics.

Figure 18: Five-sided patch on the sphere

Acknowledgement

The authors wish to thank the referees for their interesting and helpful comments which gave rise to further investigations concerning the properties of parametric representations for quadric surfaces.

References

1. **Sederberg, T. and Anderson, D.** 'Steiner Surface Patches', *IEEE Computer Graphics and Applications 5* (May 1985), pp. 23-26
2. **Piegl, L.** 'The sphere as a rational Bézier surface', *Computer Aided Geometric Design* 3 (1986), pp. 45-52
3. **Farin, G., Piper, B. and Worsey, A.J.** 'The octant of a sphere as a nondegenerate triangular Bézier patch', *Computer Aided Geometric Design* 4 (1987), pp.329-332
4. **Geise, G. and Langbecker, U.** 'Finite quadric segments with four conic boundary curves', *Computer Aided Geometric Design* 7 (1990), pp. 141-150
5. **Warren, J. and Lodha, S.** 'A Bézier Representation for Quadric Surface Patches', *Computer Aided Design* 22 (1990), pp. 574-579
6. **Boehm, W. and Hansford, D.** 'Bézier patches on Quadrics', in **Farin, G. (ed.)** *NURBS for Curve and Surface Design*, SIAM, Philadelphia (1991), pp. 1-14
7. **Fink, U.** *Biquadratische Bézier-Flächenstücke auf Quadriken*, Dissertation, Universität Stuttgart (1992)

8. **Hoschek, J. and Seemann, G.** 'Spherical Splines', *Mathematical Modelling and Numerical Analysis* 26 (1992), pp. 1-22
9. **Dietz, R., Hoschek, J. and Jüttler, B.** 'An algebraic approach to curves and surfaces on the sphere and on other quadrics', *Computer Aided Geometric Design 10* (1993)
10. **Coxeter, H.S.M.** *Projective Geometry*, Blaisdell, New York, London, Toronto (1964)
11. **Hoschek, J. and Lasser, D.** *Fundamentals of Computer Aided Geometric Design*, Peters, Wellesley (1993)
12. **Coxeter, H.S.M.** *Introduction to Geometry*, John Wiley & Sons Inc., New York (1961)
13. **Giering, O.** *Vorlesungen über höhere Geometrie*, Vieweg, Braunschweig (1982)
14. **Hlavatý, V.** *Differential Line Geometry*, P. Noordhoff, Groningen-Holland (1953), pp. 31-34
15. **Mørken, K.** 'Some identities for products and degree raising of splines', *Constructive Approximation 7* (1991), pp. 195-208
16. **Vermeulen, A.H., Bartels, R.H. and Heppler, G.R.** 'Integrating Products of B-Splines', *SIAM J. Sci. Stat. Comput.* 13 (1992), pp. 1025-1038
17. **Sabin, M.A.** 'Non-four-sided patch expressions with control points', *Computer Aided Geometric Design 1* (1985), pp. 289-290