Spatial Rational Motions and their Application in Computer Aided Geometric Design

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Abstract. Using rational motions it is possible to apply the powerful methods of Computer Aided Geometric Design to kinematical problems. The present paper discusses the so–called sweeping surfaces which are generated by the motion of a rigid profile curve through space. As the main result we derive a general construction of rational tensor–product sweeping surfaces of fixed polynomial degree. Rational surface representations prove to provide the exact description of non–trivial sweeping surfaces.

§1. Introduction

During last years, the use of rational curve and surface representations (e.g., non–uniform rational B–spline (NURBS) curves and surfaces) has been a subject of increasing interest in Computer Aided Geometric Design. Based on such representations it is possible to apply the powerful methods of CAGD to kinematical problems, e.g. from Robotics or from Computer Graphics.

Rational motions can be said to be the direct generalization of rational curves to Kinematics. Such motions have been discussed already in 1895 by Darboux [3]. They are defined by the property, that the trajectories of the points of the moving object(s) are (piecewise) rational curves. Therefore we can apply the algorithms of CAGD directly to these curves. Additionally, the surfaces which are generated by rational motions (as envelopes of moving planes or as sweeping surfaces) are rational tensor–product surfaces. Resulting from this, the use of rational motions in Kinematics supports the data exchange with CAD systems.

Rational motions can be constructed with help of a representation formula derived in [7]. Based on B–spline techniques some methods for the computer–aided design of rational motions have been developed in [8]. The motions can be described with help of a control structure which generalizes the control polygon of rational Bézier or B–spline curves. For instance, using this

Mathematical Methods in CAGD III
M. Dæhlen, T. Lyche, and L. L. Schumaker (eds.), pp. 1–10.

ISBN xxx.

control structure it is possible to formulate a simple algorithm for collision detection. In order to construct a (piecewise) rational motion from a sequence of given positions of a moving object, the interpolation and approximation problem is discussed in [8].

In the present paper we examine the construction of rational tensor–product representations of sweeping surfaces with a kinematical net of parameter lines. Such surfaces are "swept out" by the motion of a rigid profile curve through space. A remarkable illustration of the construction of a sweeping surface is given by Figure 1.12 of Bézier's preface to [5]. Note that the so–called "lofting modellers" of CAD systems are based on a similar principle of surface generation. Rational tensor–product representations of sweeping surfaces with a kinematical net of parameter lines have been discussed by Röschel at first [11] (cf. also [9]). He proved that any such representation of degree (m, n) (where n is the degree of the profile curve) is generated by a rational motion of maximal degree $2m$.

Based on the representation formula for spatial rational motions, the present paper derives a stronger result. We obtain a general construction of rational tensor–product representations of sweeping surfaces with a kinematical net of parameter lines. For instance, such representations can be constructed by interpolating a sequence of given positions of the profile curve.

§2. Spatial Motions

This section presents some notions from spatial kinematics. For a more detailed introduction the reader is referred to the textbook of Bottema and Roth [1].

Consider two coinciding Euclidean 3–spaces E resp. $\widehat{\mathrm{E}}$ which are associated with Cartesian coordinate systems. The two spaces are projectively completed by adding points at infinity which correspond to classes of parallel lines. The points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ resp. $\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \widehat{\mathbf{r}}$ of both spaces will be described with help of *homogeneous coordinate vectors*, see [6]. For instance, let $\mathbf{p} = (p_0\, p_1\, p_2\, p_3)^\top$ ($\mathbf{p} \in \mathbb{R}^4 \setminus \{\mathbf{0}\}$) be the homogeneous coordinate vector of a point from E ($p_0 \neq 0$). Then, the Cartesian coordinates $(x_p\, y_p\, z_p)$ of this point can be obtained from

$$x_p = \frac{p_1}{p_0}, \quad y_p = \frac{p_2}{p_0} \quad \text{and} \quad z_p = \frac{p_3}{p_0}. \tag{1}$$

The vectors $\mathbf{p}$ and $\lambda\mathbf{p}$ describe the same point of E for any real $\lambda \neq 0$. Homogeneous coordinate vectors with $p_0 = 0$ (but $(p_1, p_2, p_3) \neq (0, 0, 0)$) correspond to points at infinity.

The linear transformation $M : \widehat{\mathrm{E}} \to \mathrm{E} : \widehat{\mathbf{p}} \mapsto \mathbf{p} = M \cdot \widehat{\mathbf{p}}$ with

$$M = \left(\begin{array}{c|ccc} v_0 & 0 & 0 & 0 \\ \hline v_1 & & & \\ v_2 & & v_0^* U & \\ v_3 & & & \end{array}\right) \quad \text{where} \quad v_0 = v_0^*(d_0^2 + d_1^2 + d_2^2 + d_3^2) \tag{2}$$

and

$$U = \begin{pmatrix} d_0^2 + d_1^2 - d_2^2 - d_3^2 & 2(d_1 d_2 - d_0 d_3) & 2(d_1 d_3 + d_0 d_2) \\ 2(d_1 d_2 + d_0 d_3) & d_0^2 - d_1^2 + d_2^2 - d_3^2 & 2(d_2 d_3 - d_0 d_1) \\ 2(d_1 d_3 - d_0 d_2) & 2(d_2 d_3 + d_0 d_1) & d_0^2 - d_1^2 - d_2^2 + d_3^2 \end{pmatrix} \quad (3)$$

$(v_0^*, v_1, \ldots, v_3, d_0, \ldots, d_3 \in \mathbb{R},\ v_0^* \neq 0,\ (d_0, d_1, d_2, d_3) \neq (0,0,0,0))$ describes an Euclidean *spatial displacement*: the space $\widehat{\mathrm{E}}$ results from E by a translation composed with a rotation. At first, the origin of $\widehat{\mathrm{E}}$ is translated to the point $\mathbf{v} = (v_0\, v_1\, v_2\, v_3)^\top$, then a rotation around the axis with normalized direction vector $\vec{\mathbf{r}}$ and angle ϕ,

$$\cos\frac{\phi}{2} = \frac{d_0}{\sqrt{d_0^2 + d_1^2 + d_2^2 + d_3^2}}, \quad \sin\frac{\phi}{2}\,\vec{\mathbf{r}} = \frac{1}{\sqrt{d_0^2 + d_1^2 + d_2^2 + d_3^2}} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad (4)$$

($\|\vec{\mathbf{r}}\| = 1$) is applied. The four numbers $d_0, \ldots, d_3$ represent the rotational part (3) of the spatial displacement. They are called the *Euler's parameters* of this mapping, see [1]. If the 8 parameters $v_0^*, v_1, \ldots, v_3, d_0, \ldots, d_3$ are chosen as continuously differentiable functions of the time t, then we obtain a one–parameter family of spatial displacements:

$$M(t) : \widehat{\mathrm{E}} \to \mathrm{E} : \quad (\widehat{\mathbf{p}}, \mathrm{t}) \mapsto \mathbf{p}(\mathrm{t}) = \mathrm{M}(\mathrm{t})\,\widehat{\mathbf{p}}. \quad (5)$$

This family describes an Euclidean *motion* of the *moving space* $\widehat{\mathrm{E}}$ with respect to the *fixed space* E. The curve $\mathbf{p}(t) = M(t)\,\widehat{\mathbf{p}}$ represents the *trajectory* or the *path* of the point $\widehat{\mathbf{p}} \in \widehat{\mathrm{E}}$ of the moving space (cf. Fig. 1). Note that the parametric representations $M(t)\,\widehat{\mathbf{p}}$ and $\sigma(t)\,M(t)\,\widehat{\mathbf{p}}$ describe the same curve for any real function $\sigma(t) \neq 0$.

§3. Sweeping Surfaces

Let an Euclidean motion $M = M(t)$ be given. Additionally, consider a segment $\widehat{\mathbf{p}}(v) \subset \widehat{\mathrm{E}}$ of a curve in the moving space ($v \in [0,1]$). The parametric representation

$$\mathbf{y}(t,v) = \rho(t,v)\, M(t)\, \widehat{\mathbf{p}}(v) \quad (t,v) \in [0,1]^2 \quad (6)$$

describes the surface patch which is "swept out" by the motion of the rigid *profile curve* $\widehat{\mathbf{p}}(v)$ through the fixed space E (see Figures 3 and 4 and 1.12 of [5]). The use of homogeneous coordinates provides the multiplication of the right–hand side of (6) by the arbitrary real function $\rho(t,v) \neq 0$ ($(t,v) \in [0,1]^2$).

The parameter lines $v =$ const. of the surface $\mathbf{y}(t,v)$ are the trajectories of the points on the profile curve, whereas the parameter lines $t =$ const. describe the positions of the moving profile curve at the instant t, hence they are congruent. Such a surface will be called a *sweeping surface* and its parameter lines are said to form a *kinematical net*.

Note that a kinematical net of parameter lines is a relatively special parametric representation of a sweeping surface. For instance, any quadric surface is the sweeping surface of a conic section [2], but there exist many parametric representations where the parameter lines do not form a kinematical net, cf. [4].

§4. Rational Sweeping Surfaces

In Computer Aided Geometric Design, curves and surfaces are often described with help of polynomial or rational parametric representations. For instance, a rational tensor–product Bézier surface patch of degree (m, n) is given by

$$\mathbf{x}(t,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(t)\, B_j^n(v)\, \mathbf{p}_{i,j}. \tag{7}$$

The symbol $B_l^k(s) = \binom{k}{l} s^l (1-s)^{k-l}$ denotes the l–th Bernstein polynomial of degree k. The coefficients $\mathbf{p}_{i,j} \in \mathbb{R}^4$ are the homogeneous coordinates of the control points, cf. [6]. In the following we will construct parametric representations of sweeping surfaces by rational tensor–product Bézier surfaces. Rational surfaces will turn out to provide the exact description of sweeping surfaces.

Consider a a rational sweeping tensor–product Bézier surface (7) of degree (m, n), where the parameter lines are assumed to form a kinematical net. The moving profile curve $\widehat{\mathbf{p}}(v) \subset \widehat{\mathrm{E}}$ of this surface (cf. (6)) is described by the rational Bézier curve

$$\widehat{\mathbf{p}}(v) = \sum_{j=0}^{n} B_j^n(v)\, \widehat{\mathbf{q}}_j \quad v \in [0,1] \tag{8}$$

with control points $\widehat{\mathbf{q}}_j \in \widehat{\mathrm{E}}$. Then we have:

Proposition 1. *If the profile curve of the rational tensor–product sweeping surface (7) of degree (m, n) is no segment of a straight line, then the motion $M(t)$ which generates the surface (see (6)) has the following properties:*

1.) *The trajectories of the points of the moving space are rational Bézier curves.*

2.) *There exists a plane in the moving space, such that the trajectories of the points contained in it are rational Bézier curves of degree m in t.*

Proof: Resulting from (7) and (8), the definition (6) of a sweeping surface yields

$$\sum_{j=0}^{n} B_j^n(v) \sum_{i=0}^{m} B_i^m(t)\, \mathbf{p}_{i,j} = \sum_{j=0}^{n} B_j^n(v)\, \rho(t,v)\, M(t)\, \widehat{\mathbf{q}}_j. \tag{9}$$

By comparing the coefficients of the $B_j^n(v)$ on the right– and on the left–hand side we obtain immediately

$$\sum_{i=0}^{m} B_i^m(t)\, \mathbf{p}_{i,j} = N(t)\, \widehat{\mathbf{q}}_j \quad (j = 0, \ldots, n) \tag{10}$$

where $N(t) = \rho(t)M(t)$ has been set. (Note that $\rho(t,v) = \rho(t)$ because the left–hand side of (10) does not depend on v.) The profile curve (8) was assumed to be no segment of a straight line, therefore at least three non–collinear control points $\widehat{\mathbf{q}}_{j_1}, \widehat{\mathbf{q}}_{j_2}$ and $\widehat{\mathbf{q}}_{j_3}$ exist, i.e., the three homogeneous coordinate vectors $\widehat{\mathbf{q}}_{j_1}, \widehat{\mathbf{q}}_{j_2}, \widehat{\mathbf{q}}_{j_3}$ are linearly independent. Let $\widehat{\mathbf{p}} = \alpha\,\widehat{\mathbf{q}}_{j_1} + \beta\,\widehat{\mathbf{q}}_{j_2} + \gamma\,\widehat{\mathbf{q}}_{j_3}$ $(\alpha, \beta, \gamma \in \mathbb{R})$ be an arbitrary point of the plane in $\widehat{\mathrm{E}}$ which is spanned by the three points $\widehat{\mathbf{q}}_{j_1}, \widehat{\mathbf{q}}_{j_2}, \widehat{\mathbf{q}}_{j_3}$. Resulting from (10), the trajectory

$$N(t)\,\widehat{\mathbf{p}} = N(t)\,(\alpha\,\widehat{\mathbf{q}}_{j_1} + \beta\,\widehat{\mathbf{q}}_{j_2} + \gamma\,\widehat{\mathbf{q}}_{j_3}) = \sum_{i=0}^{m} B_i^m(t)\,(\alpha\,\mathbf{p}_{i,j_1} + \beta\,\mathbf{p}_{i,j_2} + \gamma\,\mathbf{p}_{i,j_3}) \tag{11}$$

of this point is a rational Bézier curve of degree m. This proves the second part of the proposition. The first part can be proved similarly by observing that the cross–product of two rational curves (in Cartesian coordinates) yields again a rational curve. ∎

In addition to the above result, Röschel has proved that the trajectories of the points of the moving space are rational curves of maximal degree $2m$. Moreover, if the profile curve is a non–planar curve, then the trajectories turn out to be rational curves of maximal degree m [11]. Based on a representation formula for motions with rational trajectories we will derive a stronger result.

§5. Rational Motions

Consider again a spatial motion $M = M(t)$ as introduced in (5).

Definition 2. *If the trajectories of the points of the moving space are rational Bézier curves (see [6]) of degree m in t, then the motion (5) is called a rational motion of degree m. Such a motion can be described by a matrix–valued function $M(t)$, where all components are polynomials of maximal degree m.*

Rational motions seem to be the appropriate tool to apply the methods of Computer Aided Geometric Design to problems from Kinematics and Robotics. Such motions can be constructed with help of the representation of spatial displacements presented in Section 2. If the 8 parameters $v_0^*(t)$, $v_i(t)$ and $d_j(t)$ are chosen as polynomials of the maximal degrees $m - 2k$, m and k, respectively $(i = 1, \ldots, 3;\ j = 0, \ldots, 3)$, where the number k satisfies $0 \leq k \leq \frac{m}{2}$, then the equations (2) and (3) yield the matrix representation of a rational motion of degree m. It is possible to construct all rational motions in this way? The answer is given by

Theorem 3. *Let a rational motion of degree m be given. Then a number k with $0 \leq k \leq \frac{m}{2}$ and 8 polynomials $v_0^*(t)$, $v_i(t)$ and $d_j(t)$ of the maximal degrees $m - 2k$, m and k, respectively, exist $(i = 1, \ldots, 3;\ j = 0, \ldots, 3)$, such that the matrix–valued polynomial obtained from (2) and (3) describes the given motion.*

This result has been derived in [7]. A detailed geometrical discussion of rational motions of order $m \leq 4$ is given in [3,12,10].

As an example, Figure 1 shows a segment of a rational motion of degree 4. The moving space $\widehat{\mathrm{E}}$ is represented by some positions of the moving unit cube, where the three pairs of opposite faces are marked by squares, triangles and crosses, respectively. Additionally, some positions of the moving coordinate system (in gray) and the trajectory of the point $(1\,1\,1\,1)^\top \in \widehat{\mathrm{E}}$ (a rational Bézier curve of degree 4) and its control polygon have been drawn.

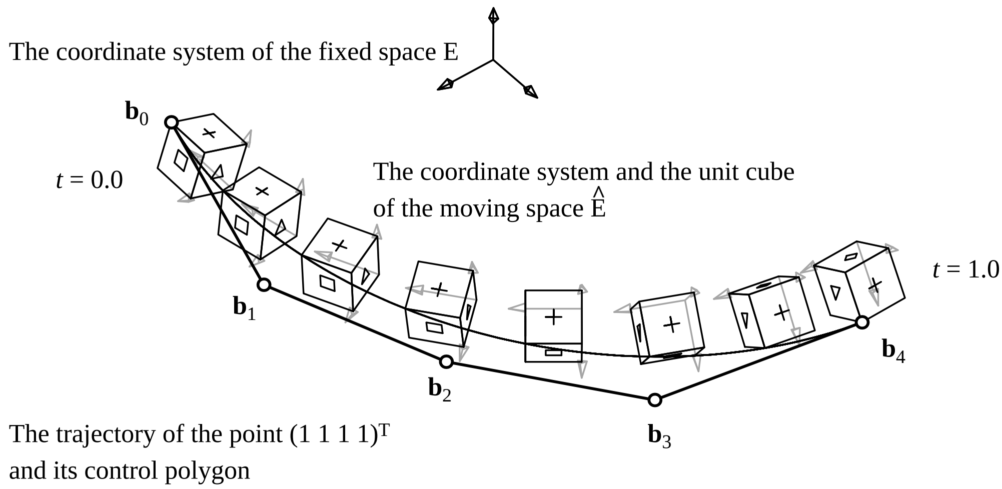

Fig. 1. A rational motion of degree 4.

§6. The Construction of Rational Sweeping Surfaces

Rational tensor–product sweeping surfaces have turned out to be generated by rational motions, cf. Proposition 1 and Definition 2. On the other hand, any rational motion can be constructed with help of the representation (2) and (3), see Theorem 3. The discussion of rational sweeping surfaces will be continued with help of the following

Lemma 4. *Let four polynomials $d_0(t), \ldots, d_3(t)$ be given. Consider the first two columns of the 3×3–matrix $U(t)$ obtained from (3). If the linear factor $(t - t_0)$ $(t_0 \in \mathbb{C})$ divides the first two columns of the matrix $U(t)$ and the polynomial $d_0(t)^2 + d_1(t)^2 + d_2(t)^2 + d_3(t)^2$, then it divides the four polynomials $d_0(t), \ldots, d_3(t)$.*

Proof: By assumption, the real number t_0 is a root of the components of the first two columns of $U(t)$ and of the polynomial $d_0(t)^2 + \ldots + d_3(t)^2$, i.e., the 5 equations

$$d_0^2 + d_1^2 + d_2^2 + d_3^2 \Big|_{t=t_0} = d_0^2 + d_1^2 - d_2^2 - d_3^2 \Big|_{t=t_0} = d_0^2 - d_1^2 + d_2^2 - d_3^2 \Big|_{t=t_0} = 0 \tag{12}$$

and

$$d_1\, d_2 + d_0\, d_3 \Big|_{t=t_0} = d_1\, d_2 - d_0\, d_3 \Big|_{t=t_0} = 0 \tag{13}$$

hold for $t = t_0$. From (12) we obtain immediately

$$d_1^2 \Big|_{t=t_0} = d_2^2 \Big|_{t=t_0} = -d_0^2 \Big|_{t=t_0} , \tag{14}$$

whereas (13) yields

$$d_0\, d_3 \Big|_{t=t_0} = d_1\, d_2 \Big|_{t=t_0} = 0. \tag{15}$$

Therefore, the equations

$$d_0 \Big|_{t=t_0} = d_1 \Big|_{t=t_0} = d_2 \Big|_{t=t_0} = d_3 \Big|_{t=t_0} \tag{16}$$

have to be satisfied. This proves the assertion. ■

As an immediate consequence we have:

Corollary 5. *Consider a rational motion $M = M(t)$ obtained from (2) and (3) by choosing the parameters v_0^*, v_1, v_2, v_3 and $d_0, \ldots, d_3$ as polynomials. If an arbitrary polynomial $\xi = \xi(t)$ divides the first three columns of the matrix $M(t)$, then it also divides the whole matrix.*

Proof: Let $\xi(t) = \alpha(t) \cdot \beta(t)$, where the polynomial $\alpha(t)$ divides $v_0^*(t)$ and the polynomial $\beta(t)$ divides $d_0(t)^2 + \ldots + d_3(t)^2$. (This factorization is possible as $\xi(t)$ divides the polynomial $v_0(t)$, cf. (2)). Resulting from Lemma 4, $\beta(t)$ divides the four polynomials $d_0(t), \ldots, d_3(t)$, thus it is a divisor of all components of the matrix $U(t)$, cf. (3). Hence, the polynomial $\xi(t)$ divides the whole matrix $M(t)$. ■

Based on this corollary we are able to discuss rational motions $M(t)$, where the trajectories of the points of a plane are of maximal degree m in t:

Proposition 6. *Consider a rational motion $M = M(t)$. If there exists a plane in the moving space, such that the trajectories of the points contained in it are rational Bézier curves of maximal degree m in t, then the motion $M(t)$ is a rational motion of maximal degree m.*

Proof: Let without loss of generality the trajectories of the points of the plane $\hat{x}_3 = 0$ be rational Bézier curves of maximal degree m in t. The motion can be described by a matrix–valued polynomial $M(t)$, where the components of this matrix do not have any common divisors (i.e., $\gcd(m_{i,j}(t))_{i,j=0,1,2,3} = 1$). Because of Corollary 5, the components of the first three columns of $M(t)$ do not have any common divisors. Therefore, the first three columns of $M(t)$ are of maximal degree m in t, as the trajectories of the points of the plane $\hat{x}_3 = 0$ were assumed to be rational Bézier curves of maximal degree m in t. Resulting from Theorem 3, (2) and (3), the motion $M(t)$ is a rational motion of maximal degree m. ■

Now we are ready to state the main result of this section:

Theorem 7. *If the profile curve of the rational tensor–product sweeping surface (7) of degree (m, n) is no segment of a straight line (i.e., if the surface (7) is no segment of a ruled surface), then this surface is generated by a rational motion of maximal degree m. Therefore, the parametric representation of the sweeping surface can be constructed from*

$$\mathbf{x}(t, v) = M(t) \cdot \widehat{\mathbf{p}}(v), \tag{17}$$

where $M(t)$ is a rational motion of degree m (see Theorem 3) and $\widehat{\mathbf{p}}(v)$ describes the moving profile curve (cf. (8)).

The proof results from Propositions 1 and 6. With help of this theorem we can easily construct all rational tensor–product sweeping surfaces of the (fixed) degree (m, n).

From the representation formula for rational motions (Theorem 3, (2) and (3)) and from the above theorem it is obvious, that *polynomial* (integral) sweeping tensor–product Bézier surfaces with a kinematical net of parameter lines can only be constructed by choosing the parameters $v_0^*(t)$, $d_0(t)$, $d_1(t)$, $d_2(t)$, $d_3(t)$ as real constants. Thus, such surfaces are generated by rational motions with constant rotational part, i.e., any two positions of the moving profile curve are related by a translation. Therefore we have: *Polynomial tensor product sweeping surfaces with a kinematical net of parameter lines are translational surfaces,* i.e., they are generated by translational motions. As another advantage of rational surface representations, these surfaces support the exact description of non–trivial sweeping surfaces.

§7. Interpolation by Rational Sweeping Surfaces

Finally we will outline a method for interpolation by rational tensor–product sweeping surfaces. Let $l + 1$ positions of the profile curve $\widehat{\mathbf{p}}(v) \subset \widehat{\mathrm{E}}$ be given. These positions are described by the $l + 1$ spatial displacements Q_i $(i = 0, \ldots, l)$, cf. (2). Additionally, a sequence of parameter values (instants) t_i which correspond to the positions of the profile curve is assumed to be known. The position of the profile curve at the instant $t = t_i$ is given by its image under the spatial displacement Q_i. If the parameters t_i are unknown, then they can be estimated from the distances between the given positions of the profile curve, cf. [8].

As an example, Figure 2 shows 4 given positions of the profile curve, where the profile curve is a segment of a cubic rational Bézier curve. The spatial displacements $Q_0 \ldots, Q_3$ are represented by the positions of the unit cube, where its faces are marked analogously to Figure 1. The profile curve is contained in the plane of the ⋄–marked face of the unit cube. An equidistant distribution of the parameters t_i has been chosen.

We will construct a rational tensor–product sweeping surface which interpolates the given positions of the profile curve. This surface is constructed in two steps: At first, the generating motion $M = M(t)$ of the sweeping surface

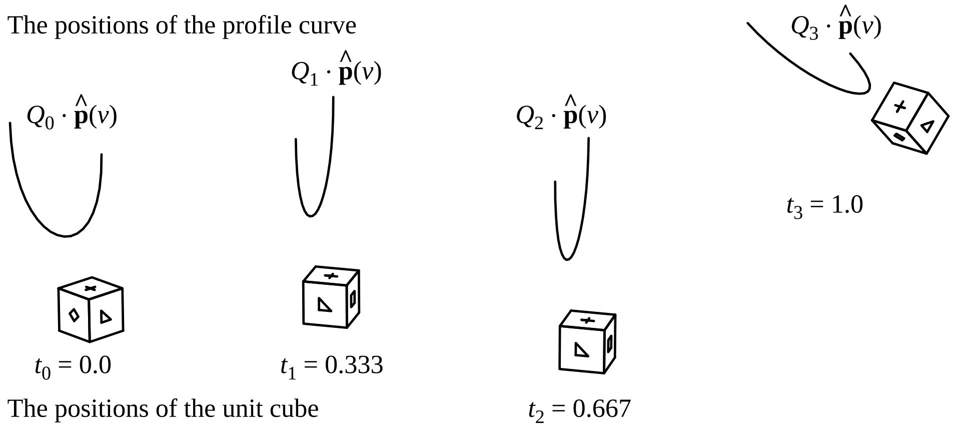

Fig. 2. The given positions of the moving profile curve.

is found by interpolating the given spatial displacements. This motion has to satisfy the interpolation conditions

$$M(t_i) = \lambda_i Q_i \quad (i = 0, \ldots, n), \tag{18}$$

where the positive real factors λ_i can be used as design parameters. For details of the method the reader is referred to [8], where interpolation by spatial (piecewise) rational motions has been discussed thoroughly.

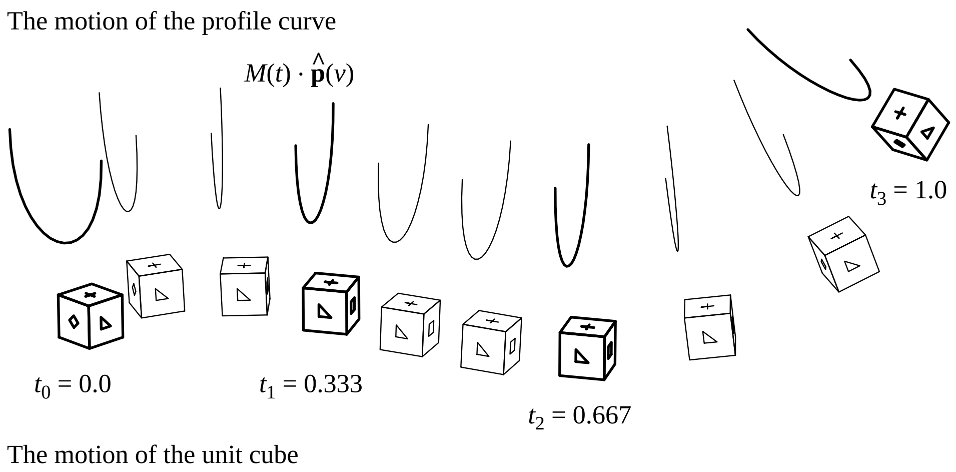

Fig. 3. The interpolating motion.

In the second step, the interpolating sweeping surface is found by moving the profile curve through space, i.e., from $\mathbf{x}(t, v) = M(t) \cdot \widehat{\mathbf{p}}(v)$.

The last two figures illustrate the construction of an interpolating sweeping surface from the given positions of Figure 2. Figure 3 shows the interpolating motion which is represented by some positions of the profile curve and of moving unit cube (similar to Fig. 2). The resulting rational tensor–product representation of degree $(6, 3)$ of the constructed sweeping surface (in gray) and its control points are drawn in Figure 4.

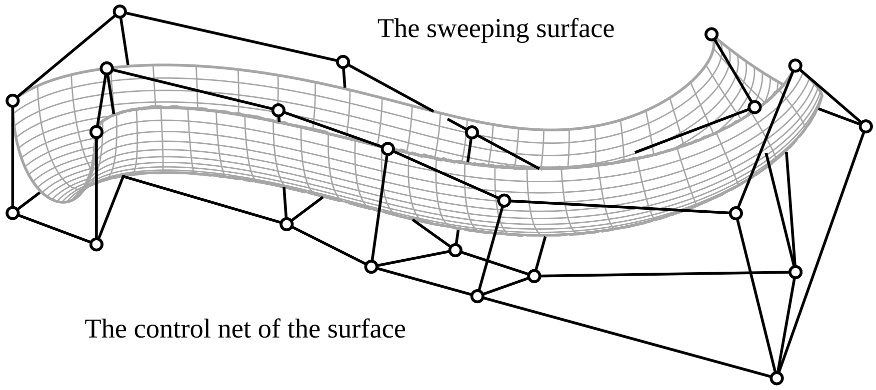

Fig. 4. The rational tensor–product sweeping surface.

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