Triangular Bézier surface patches with a linear normal vector field

B. Jüttler
University of Technology, Darmstadt

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Abstract

The article is devoted to triangular Bézier surface patches with a linear normal vector field (LN surfaces). These surface patches are shown to have rational offset surfaces. This fact generalizes the rational offset property of parabolas to the surface case. LN surface patches may serve as an alternative to surfaces with Pythagorean normals (the so–called PN surfaces, see [17, 18]). We present a construction for triangular LN surface patches from Hermite boundary data. The construction is suitable for building G^1 surface splines.

1 Introduction

The description and the design of offsets to curves and surfaces has attracted a great deal of attention from the geometric design community. Offsets arise in the context of the numerical control of milling machines and layered manufacturing. Many methods rely on approximations to the exact offsets, e.g. [1]. Recently, a number of constructions for rational curves and surfaces with rational offsets have emerged. Using these techniques one may develop curve and surface schemes that make it possible to represent both a certain shape and its offsets exactly within a CAD system.

At the beginning of these developments, the notion of Pythagorean hodograph (PH) curves have been introduced by Farouki. These curves form a sub–class of integral Bézier curves; they are distinguished by having rational offset curves and a polynomial arc length function. Using complex calculus, constructions for PH spline curves from various input data has been developed, e.g. [9, 10].

Another approach has been used by Pottmann [11, 18] in order to find a construction for rational PH curves. His approach is based on the dual representation of a planar curve (resp. surface) as the envelope of its tangent lines (resp. planes). Using this approach, it is also possible to study rational surfaces with rational offsets, the so-called PH or PN (Pythagorean normal vector, see [17]) surfaces [18]. Based on a dual representation of the unit sphere, an elegant construction of the dual control structure for this class of surfaces has been derived by Pottmann [18]. The recent Ph.D. thesis by Peternell [17] extends the dual approach to PH curves and surfaces by introducing concepts from Laguerre geometry. In this geometry, offsetting a shape is an intrinsic operation; hence Laguerre geometry is particularly well suited for dealing with offset curves and surfaces.

The dual approach to offsets produces rational curves and surfaces. In consequence of working with the dual representation, singularities (like cusps which are dual to inflections) and points at infinity may cause considerable problems. Also, the dual control structure seems to be very sensitive to small perturbations of the components. So far, there seems to be no construction scheme for PN spline surfaces available which is suitable for practical implementations.

In the present paper we introduce a new class of surfaces with rational offsets. Triangular Bézier surface patches with a linear field of normal vectors (the so–called LN surface patches) are shown to have rational offset surfaces. This fact generalizes the rational offset property of parabolas [12] to the surface setting. Unlike PN surfaces, the class of LN surface patches is invariant under affine mappings.

As a major difference to the dual approach to PH curves and surfaces, we use the distinguishing property of the linear normal field as a linear constraint onto the linear space of integral triangular Bézier patches. In addition, the LN surface is an integral patch, but its offset surfaces are truly rational surfaces. That is, we design an integral patch which is embedded into a family of rational offset surfaces. We think that there are good prospects that LN surface splines can be developed into a tool which is suitable for practical implementations, because our methods are simply based on integral patches and linear spaces of functions.

In the first section of this article we give a brief discussion of curves with a linear normal vector field. This section is intended to serve as motivation for the following section which is devoted to LN surface patches. Finally we present a construction of LN patches from Hermite boundary data which illustrates the potential of the new surface scheme.

2 Motivation: Curves with linear normals

Consider an integral (i.e. a polynomial) quadratic curve segment,

$$\mathbf{x}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2, \qquad t \in [0,1], \tag{1}$$

in the plane $\mathbb{R}^2$. The curve is assumed to be given by its monomial representation (with respect to the power basis 1, t, t^2) with certain coefficient vectors $\mathbf{p}_0, \ldots, \mathbf{p}_2 \in \mathbb{R}^2$. Both components of $\mathbf{x}(t)$ are quadratic polynomials of the parameter t. For instance, any quadratic Bézier curve (see [13]) has such a representation.

It is a well–known fact that integral quadratic curves are either segments of lines or of parabolas. In order to find the offset (or parallel) curves to the quadratic (1), we compute the first derivative vector $\dot{\mathbf{x}}(t)$ and the non–normalized normal vector $\vec{\mathbf{N}}(t)$,

$$\dot{\mathbf{x}}(t) = \mathbf{p}_1 + 2\,\mathbf{p}_2\,t, \quad \text{and} \quad \vec{\mathbf{N}}(t) = \mathbf{p}_1^\perp + 2\,\mathbf{p}_2^\perp\,t. \tag{2}$$

Here, we denote with $\mathbf{x}^\perp$ the vector which is obtained after rotating $\mathbf{x}$ by 90 degrees,

$$\mathbf{x}^\perp = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{3}$$

The offset (or parallel) curve of the quadratic (1) at distance d has the parametric representation

$$\mathbf{x}_d(t) = \mathbf{x}(t) + \frac{d}{\|\vec{\mathbf{N}}(t)\|}\,\vec{\mathbf{N}}(t) \tag{4}$$

with the norm

$$\|\vec{\mathbf{N}}(t)\| = \sqrt{4\,(\mathbf{p}_2, \mathbf{p}_2)\,t^2 + 4\,(\mathbf{p}_1, \mathbf{p}_2)\,t + (\mathbf{p}_1, \mathbf{p}_1)} \tag{5}$$

of the normal vector. Here, $(.,.)$ (resp. $[.,.]$) is the standard inner product (resp. exterior product, i.e., the determinant) of two vectors from $\mathbb{R}^2$. Clearly, the coordinates of offset curves (4) are generally not rational functions of the curve parameter t, owing to the square root in the denominator. The offset curves to a parabola, however, are known to be rational curves, see [12, 15].

In order to find a rational parametric representation of the offset curves, we apply the substitution $t(s) = \tau(s)/\sigma(s)$ to the offset curves, where $\tau(s)$ and $\sigma(s)$ are certain rational functions of the new curve parameter s. This gives

$$\|\vec{\mathbf{N}}(t(s))\| = \sqrt{K(\sigma,\tau)}\;/\;|\sigma(s)|$$

$$\text{with} \quad K(\sigma,\tau) = 4\,(\mathbf{p}_2, \mathbf{p}_2)\,\tau(s)^2 + 4\,(\mathbf{p}_1, \mathbf{p}_2)\,\tau(s)\sigma(s) + (\mathbf{p}_1, \mathbf{p}_1)\,\sigma(s)^2.$$

Now consider the curve $K(\sigma,\tau) = 1$ in the $\sigma\tau$–plane. It is an implicit quadratic curve, i.e., a conic section. If the coefficient vectors $\mathbf{p}_1$, $\mathbf{p}_2$ are linearly independent (i.e., $[\mathbf{p}_1, \mathbf{p}_2] \neq 0$), then the conic turns out to be an ellipse which is centered at $(0,0)$. (Otherwise the quadratic (1) degenerates into a straight line segment.) Thus, one may easily find quadratic polynomials $\tau_0(s)$, $\sigma_0(s)$ and $\rho(s)$ such that $\tau = \tau_0/\rho$ and $\sigma = \sigma_0/\rho$ is a rational parametric representation of the ellipse, i.e. the identity

$$K(\,\sigma_0(s)/\rho(s)\,, \tau_0(s)/\rho(s)\,) \equiv 1, \qquad s \in \mathbb{R}, \tag{6}$$

holds. For more information concerning rational parameterizations of conics the reader should consult [7, 13].

Owing to (6) we have $\|\vec{\mathbf{N}}(t(s))\| = |\rho(s)/\sigma_0(s)|$. Thus, by applying the substitution $t(s) = \tau(s)/\sigma(s)$ to the offsets (4) of the quadratic we obtain rational parametric representations with the new parameter s for these offset curves.

As the essential ingredient of this construction for rational offsets, we have exploited the fact that the quadratic curve has a *linear normal vector*. One may immediately apply the same idea to other curves with linear normals. For instance, consider the cubic

$$\mathbf{x}(t) = \mathbf{q}_0 + \mathbf{q}_1\, t + \mathbf{q}_2\, t^2 + \mathbf{q}_3\, t^3, \qquad t \in [0,1], \tag{7}$$

in the plane $\mathbb{R}^2$. Again, the curve is assumed to be given in monomial representation with the coefficient vectors $\mathbf{q}_0, \ldots, \mathbf{q}_3 \in \mathbb{R}^2$. Generally, its first derivative vector has quadratic components. However, if the first derivative vanishes for a certain parameter value $t = t_0$, i.e. if the cubic has a cusp at $t = t_0$, then the first derivative factors into

$$\dot{\mathbf{x}}(t) = (t - t_0)\ (\mathbf{p}_1 + 2\,\mathbf{p}_2\, t) \tag{8}$$

with certain coefficient vectors $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^2$. Hence, under this additional assumption, the cubic has the linear normal vector $\vec{\mathbf{N}}(t)$, see (2). Now we can use again the above construction in order to find rational parametric representations of the offset curves. More generally, the construction can be applied to polynomial curves of degree n with $n-2$ cusps. Unlike the class of Pythagorean–hodograph curves (see the related publications by Farouki, e.g. [9]), the class of all curves with a linear normal vector is invariant with respect to affine mappings.

3 Triangular LN Bézier surface patches

Now we transfer the observation from the previous section to the surface case. In the sequel we consider an integral triangular Bézier surface patch of degree n,

$$\mathbf{x}(u,v,w) = \sum_{\substack{i,j,k\,\geq\,0\\ i+j+k\,=\,n}} \mathbf{p}_{i,j,k}\ B^n_{i,j,k}(u,v,w) \qquad u,v,w \geq 0,\quad u+v+w = 1, \tag{9}$$

see [6, 13]. Its parameters u, v, w can be seen as the barycentric coordinates with respect to some domain triangle $\triangle \subset \mathbb{R}^2$. They will be referred to as the barycentric parameters of the triangular patch. The coefficients $\mathbf{p}_{i,j,k}$ are the control points of the surface patch. The blending functions $B^n_{i,j,k}(u,v,w) = \frac{n!}{i!j!k!}\ u^i\, v^j\, w^k$ are the bivariate Bernstein polynomials defined over the domain triangle $\triangle$.

The first directional derivatives of the triangular Bézier patch are polynomials of degree $n-1$. We recall the formulas for the first derivatives in the three directions which are parallel to the edges of the domain triangle,

$$\begin{aligned}
\mathbf{x}_1(u,v,w) &= \frac{\partial}{\partial t}\mathbf{x}(u,v-t,w+t)\Big|_{t=0} = n \sum_{\substack{i,j,k\geq 0\\ i+j+k=n-1}} \Delta_1\mathbf{p}_{i,j,k}\; B^{n-1}_{i,j,k}(u,v,w),\\
\mathbf{x}_2(u,v,w) &= \frac{\partial}{\partial t}\mathbf{x}(u+t,v,w-t)\Big|_{t=0} = n \sum_{\substack{i,j,k\geq 0\\ i+j+k=n-1}} \Delta_2\mathbf{p}_{i,j,k}\; B^{n-1}_{i,j,k}(u,v,w),\\
\mathbf{x}_3(u,v,w) &= \frac{\partial}{\partial t}\mathbf{x}(u-t,v+t,w)\Big|_{t=0} = n \sum_{\substack{i,j,k\geq 0\\ i+j+k=n-1}} \Delta_3\mathbf{p}_{i,j,k}\; B^{n-1}_{i,j,k}(u,v,w),
\end{aligned} \tag{10}$$

with the difference vectors

$$\begin{aligned}
&\Delta_1\mathbf{p}_{i,j,k} = \mathbf{p}_{i,j,k+1} - \mathbf{p}_{i,j+1,k}, \quad \Delta_2\mathbf{p}_{i,j,k} = \mathbf{p}_{i+1,j,k} - \mathbf{p}_{i,j,k+1},\\
&\Delta_3\mathbf{p}_{i,j,k} = \mathbf{p}_{i,j+1,k} - \mathbf{p}_{i+1,j,k}, \;\; i,j,k \geq 0, \;\; i+j+k = n-1.
\end{aligned} \tag{11}$$

In addition, let

$$\Delta_l\mathbf{b}_{i,j,k} = 0 \text{ whenever } i<0 \text{ or } j<0 \text{ or } k<0, \quad l = 1,2,3. \tag{12}$$

The above three derivative vectors are linearly dependent,

$$\mathbf{x}_1(u,v,w) + \mathbf{x}_2(\ldots) + \mathbf{x}_3(\ldots) = 0 \;\text{ resp. }\; \Delta_1\mathbf{p}_{i,j,k} + \Delta_2\mathbf{p}_{i,j,k} + \Delta_3\mathbf{p}_{i,j,k} = 0. \tag{13}$$

As in the previous section we denote with $(.,.)$ the standard inner product of vectors in $\mathbb{R}^3$.

Definition. *The triangular Bézier surface (9) will be called a* LN surface patch *(a surface with a* ***l****inear* ***n****ormal vector), if there are three vectors* $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$, $\vec{\mathbf{n}}_3 \in \mathbb{R}^3$ *such that the three inner products between the first derivative vectors (10) and the normal vector field*

$$\vec{\mathbf{N}}(u,v,w) = u\,\vec{\mathbf{n}}_1 + v\,\vec{\mathbf{n}}_2 + w\,\vec{\mathbf{n}}_3 \tag{14}$$

vanish identically, i.e., if the three equations

$$\Big(\; \mathbf{x}_l(u,v,w),\;\; \vec{\mathbf{N}}(u,v,w) \;\Big) \equiv 0, \qquad l = 1,2,3, \tag{15}$$

hold for all coordinates u,v,w *with* $u+v+w=1$. *At least one of the three vectors* $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$, $\vec{\mathbf{n}}_3$ *is assumed to be not the null vector.*

Thus, a LN surface patch has a linear field (14) of (non–normalized) normal vectors. If the three vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$, $\vec{\mathbf{n}}_3$ are given, then the three conditions (15) lead to a system of linear equations for the control points $\mathbf{p}_{i,j,k}$. By comparing the coefficients we obtain after some straightforward calculations the $3\binom{n+2}{2}$ linear equations

$$\begin{aligned} \tfrac{i}{n}(\vec{\mathbf{n}}_1, \Delta_l \mathbf{p}_{i-1,j,k}) + \tfrac{j}{n}(\vec{\mathbf{n}}_2, \Delta_l \mathbf{p}_{i,j-1,k}) + \tfrac{k}{n}(\vec{\mathbf{n}}_3, \Delta_l \mathbf{p}_{i,j,k-1}) = 0 \\ i,j,k \geq 0,\ i+j+k=n,\ l=1,2,3, \end{aligned} \tag{16}$$

see also (12). Owing to the dependency (13) between the first directional derivatives, it suffices to use the equations for $l = 1$ and $l = 2$. Hence, we obtain from (16) a system of $2\binom{n+2}{2}$ equations which are sufficient conditions for a LN surface patch.

Clearly, the orthogonality conditions (15) are rather restrictive constraints on the possible surfaces. Nevertheless, LN surface patches still have enough degrees of freedom to describe interesting shapes. Figure 1 shows two examples of triangular LN Bézier surfaces of degree 6, a convex patch (left) and a non–convex one (right). In addition to the surfaces, both pictures show the control polygons (dashed lines) and some level curves z = const. (solid lines). Also, the normal vectors at the three corner points have been drawn (thick light–grey lines).

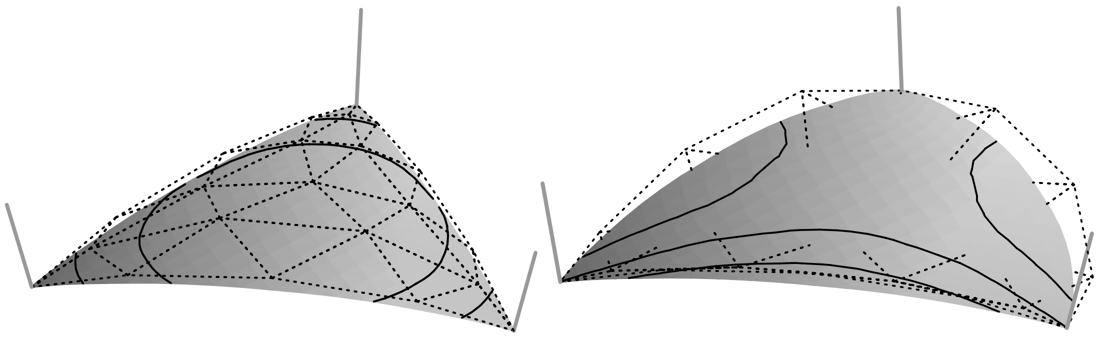

Figure 1: A convex and a non–convex LN surface patch

The surface patches have been obtained from the construction which is described in the next section. In order to avoid compatibility problems, and in order to decouple the computation of the boundary curves from the remaining inner control points, this construction uses triangular patches with degenerate points at the three vertices; all first derivative vectors at these points are zero. However, the surfaces have a well defined tangent plane everywhere, as the neighbouring control points at the vertices satisfy certain compatibility conditions. Thus, both patches in Figure 1 are regular surfaces, despite the singularly parameterized vertices.

The class of LN surface patches generalizes the rational offset property of LN curves.

Theorem. *Triangular LN surface patches have rational offsets.*

Proof. The offset surface to the LN surface surface patch (9) at a certain distance d has the parametric representation

$$\mathbf{x}_d(u,v,w) = \mathbf{x}(u,v,w) + \frac{d}{\|\vec{\mathbf{N}}(u,v,w)\|}\,\vec{\mathbf{N}}(u,v,w), \tag{17}$$

see (14). In order to find a rational representation of the offset surface, we introduce the new barycentric parameters r, s, t, $r+s+t=1$, by substituting

$$\begin{aligned} u(r,s,t) &= \rho(r,s,t) \;/\; (\,\rho(r,s,t)+\sigma(r,s,t)+\tau(r,s,t)\,),\\ v(r,s,t) &= \sigma(r,s,t) \;/\; (\,\rho(r,s,t)+\sigma(r,s,t)+\tau(r,s,t)\,),\\ w(r,s,t) &= \tau(r,s,t) \;/\; (\,\rho(r,s,t)+\sigma(r,s,t)+\tau(r,s,t)\,), \end{aligned} \tag{18}$$

where

$$\rho(r,s,t) = \frac{\rho_0(r,s,t)}{\xi(r,s,t)},\quad \sigma(r,s,t) = \frac{\sigma_0(r,s,t)}{\xi(r,s,t)},\quad \tau(r,s,t) = \frac{\tau_0(r,s,t)}{\xi(r,s,t)}. \tag{19}$$

Here, ρ_0, σ_0, τ_0, and ξ are certain polynomials of the new barycentric parameters r, s, t. They can be chosen as bivariate polynomials in Bernstein–Bézier representation with respect to a domain triangle, see [6, 13].

Consider the norm of the linear normal vector $\vec{\mathbf{N}}(u,v,w)$. The above substitution leads to

$$\|\,\vec{\mathbf{N}}(u(r,s,t),v(\ldots),w(\ldots))\,\| = \frac{\sqrt{K(\rho(r,s,t),\sigma(\ldots),\tau(\ldots))}}{|\rho(r,s,t)+\sigma(\ldots)+\tau(\ldots)|} \tag{20}$$

with the quadratic form

$$K(\rho,\sigma,\tau) = (\,\rho\;\sigma\;\tau\,)\begin{pmatrix} (\vec{\mathbf{n}}_1,\vec{\mathbf{n}}_1) & (\vec{\mathbf{n}}_1,\vec{\mathbf{n}}_2) & (\vec{\mathbf{n}}_1,\vec{\mathbf{n}}_3)\\ (\vec{\mathbf{n}}_1,\vec{\mathbf{n}}_2) & (\vec{\mathbf{n}}_2,\vec{\mathbf{n}}_2) & (\vec{\mathbf{n}}_2,\vec{\mathbf{n}}_3)\\ (\vec{\mathbf{n}}_1,\vec{\mathbf{n}}_3) & (\vec{\mathbf{n}}_2,\vec{\mathbf{n}}_3) & (\vec{\mathbf{n}}_3,\vec{\mathbf{n}}_3) \end{pmatrix}\begin{pmatrix}\rho\\ \sigma\\ \tau\end{pmatrix}. \tag{21}$$

If the three vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$ and $\vec{\mathbf{n}}_3$ are linearly independent, then this quadratic form is positive definite, as it satisfies Sylvester's criterion, cf. [3, Section 8.3, Thm. 6]. Thus, the equation

$$K(\rho,\sigma,\tau) = 1 \tag{22}$$

describes an ellipsoid in $\rho\sigma\tau$–space centered at $(0,0,0)$. If only two of the three vectors are linearly independent, then this ellipsoid degenerates into an elliptic cylinder. Finally, if any two of the three vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$ and $\vec{\mathbf{n}}_3$ are linearly dependent, then we get a double plane. In this case, however, the LN surface is simply a planar surface patch, and its offsets are obviously rational.

In the first two cases one may construct a rational parametric representation of the quadric surface (22). That is, we may find polynomials ρ_0, σ_0, τ_0, and ξ

of the barycentric parameters r, s, t such that the identity (22) is satisfied for all barycentric parameters. The coordinates

$$\begin{pmatrix} \rho(r,s,t) \\ \sigma(r,s,t) \\ \tau(r,s,t) \end{pmatrix} = \begin{pmatrix} \rho_0(r,s,t)/\xi(r,s,t) \\ \sigma_0(r,s,t)/\xi(r,s,t) \\ \tau_0(r,s,t)/\xi(r,s,t) \end{pmatrix}$$

describe nothing but a rational triangular Bézier patch with the barycentric parameters r, s, t on the quadric (22) in $\rho\sigma\tau$–space. Such patches can be constructed with the help of the stereographic projection [2] or using the more sophisticated techniques described in [5].

In order to cover the whole LN surface patch with the reparameterization (18), one has to find a triangular patch which covers the octant of the ellipsoid (22) that is bounded by the coordinate planes. For instance, a quartic rational triangular patch that describes an octant of the unit sphere is presented in [8].

Owing to (22), the substitution (18) leads to

$$\| \vec{\mathbf{N}}(u(r,s,t), v(\ldots), w(\ldots)) \| = \frac{|\xi(r,s,t)|}{|\rho_0(r,s,t) + \sigma_0(\ldots) + \tau_0(\ldots)|}. \tag{23}$$

Hence, combining this result with (17) we obtain rational parametric representations of the offset surfaces. □

The so called Pythagorean–hodograph (PH; also called Pythagorean normal vector, PN) surfaces form another class of surfaces with rational offsets, see [17, 18]. These surfaces are not invariant under affine mappings; in general, the affine image of a PH surface is not again a PH surface. Clearly, one would not expect to have such an invariance property, as affine transformations and the offsetting operation do not commute. For LN surface patches, however, there is the following result:

Proposition. *The class of LN surface patches is invariant under affine mappings.*

Proof. Consider a regular affine transformation of $\mathbb{R}^3$,

$$\mathbf{x} \mapsto \vec{\mathbf{u}} + A\,\mathbf{x}, \tag{24}$$

with the translation vector $\vec{\mathbf{u}} \in \mathbb{R}^3$ and the non–singular real 3×3–matrix A. The image $\vec{\mathbf{u}} + A\,\mathbf{x}(u,v,w)$ of the triangular LN surface patch (9) has the first derivative vectors $A\,\mathbf{x}_l(u,v,w)$, $l = 1,2,3$. Thus, the vectors $A^{-1}\vec{\mathbf{N}}(u,v,w)$ (see (14)) form a linear normal vector field of the image surface, as

$$(A\,\mathbf{x}_l, A^{-1}\vec{\mathbf{N}}) = (A\,\mathbf{x}_l)^\top (A^{-1}\vec{\mathbf{N}}) = \mathbf{x}_l^\top \vec{\mathbf{N}} = (\mathbf{x}_l, \vec{\mathbf{N}}) \tag{25}$$

holds. □

In order to develop LN surfaces into a tool which is suitable for practical implementations, it will be most important to find criteria that guarantee the regularity of the surface patches. (The triangular patch (9) is said to be regular at a point, if the first derivatives $\mathbf{x}_1(u,v,w)$ and $\mathbf{x}_2(u,v,w)$ are linearly independent.) A detailed discussion of regularity criteria is beyond the scope of the present paper. However, we outline an idea that may help to develop such criteria.

Once a LN surface has been constructed, then the cross product of the first derivatives $\mathbf{x}_1(u,v,w)$ and $\mathbf{x}_2(u,v,w)$ is always linearly dependent on the linear normal vector field $\vec{\mathbf{N}}(u,v,w)$. Thus, there is a bivariate function $\phi(u,v,w)$ such that

$$\mathbf{x}_1(u,v,w) \times \mathbf{x}_2(u,v,w) = \phi(u,v,w)\,\vec{\mathbf{N}}(u,v,w) \tag{26}$$

holds for all barycentric parameters u,v,w. If the vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$, $\vec{\mathbf{n}}_3$ are linearly independent, then $\phi(u,v,w)$ is guaranteed to be a bivariate polynomial. This can easily be concluded by comparing the zeros of the three coordinates of $\vec{\mathbf{N}}(u,v,w)$ and $\mathbf{x}_1(u,v,w) \times \mathbf{x}_2(u,v,w)$. If the polynomial $\phi(u,v,w)$ is strictly positive in the interior of the domain triangle, and if the linear normal vector field does not vanish (this is automatically guaranteed if the three vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$ and $\vec{\mathbf{n}}_3$ are linearly independent), then the LN surface patch (9) is regular at all inner points. Currently we are trying to develop a construction of LN spline surfaces of relatively low degree, where it is possible to compute the polynomials ϕ explicitly as functions of the input data, with the help of some computer algebra tools.

The observation (26) may also be helpful in order to find geometric criteria for LN surface patches. Generally, the normal vector field of a quadratic triangular Bézier patch (9) has degree 2. However, if the patch degenerates along a line in the parameter domain (for instance, if the control points of one of the boundaries are identical), then the normal factors into a linear vector field (14), multiplied with a linear polynomial $\phi(u,v,w)$. The triangular patch is then a LN surface. See also [4] for a geometric classification of quadratic triangular Bézier surfaces.

4 A construction of triangular LN patches from Hermite boundary data

In the final section we present a construction of triangular LN surface patches from given boundary data. The construction can be used in order to build G^1 surface splines.

Consider three points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3 \in \mathbb{R}^3$ with associated normal vectors $\vec{\mathbf{n}}_1$, $\vec{\mathbf{n}}_2$, $\vec{\mathbf{n}}_3$. We want to find a triangular LN surface patch with the three vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ which has the linear normal vector field (14). Our construction proceeds in two steps.

1.) Firstly we construct the three boundary curves of the LN patch. In order to make the construction suitable for building G^1 surface splines, the control points of each boundary are to be independent of the opposite vertex and normal vector. For instance, the boundary which joins $\mathbf{v}_1$ and $\mathbf{v}_2$ should solely depend on the data $\mathbf{v}_1$, $\vec{\mathbf{n}}_1$, $\mathbf{v}_2$, and $\vec{\mathbf{n}}_2$. In addition, the construction has to be symmetric in these data. If these requirements are satisfied, then any two LN patches sharing the boundary data $\mathbf{v}_1$, $\vec{\mathbf{n}}_1$, $\mathbf{v}_2$, and $\vec{\mathbf{n}}_2$ will automatically meet with G^1 continuity along the common boundary.

2.) In the second step we fill in a patch having the prescribed normal vector field (14). In order to find this patch, one has to take care of certain compatibility conditions, both at the vertices and along the boundaries, see below.

Before describing the steps of the construction in somewhat more detail, we derive two types of compatibility conditions.

4.1 Compatibility at the vertices

The triangular LN surface patch satisfies the conditions $(\vec{\mathbf{N}}, \mathbf{x}_1) \equiv 0$ and $(\vec{\mathbf{N}}, \mathbf{x}_2) \equiv 0$. Differentiating these identities we obtain

$$(\vec{\mathbf{N}}_2, \mathbf{x}_1) + (\vec{\mathbf{N}}, \mathbf{x}_{12}) \equiv (\vec{\mathbf{N}}_1, \mathbf{x}_2) + (\vec{\mathbf{N}}, \mathbf{x}_{12}) \equiv 0,$$

hence

$$(\vec{\mathbf{N}}_1, \mathbf{x}_2) = (\vec{\mathbf{N}}_2, \mathbf{x}_1). \tag{27}$$

Here, the lower indices denote the directional derivatives in directions which are parallel to the edges of the domain triangle, see (10).

Consider the vertex $\mathbf{v}_3 = \mathbf{x}(0,0,1)$ of the LN surface patch. The directional derivative $\mathbf{x}_1(0,0,1)$ (resp. $\mathbf{x}_2(0,0,1)$) is the first derivative vector of the boundary joining the vertices $\mathbf{v}_2$ (resp. $\mathbf{v}_1$) and $\mathbf{v}_3$. On the other hand,

$$\vec{\mathbf{N}}_1 = \vec{\mathbf{n}}_3 - \vec{\mathbf{n}}_2 \text{ and } \vec{\mathbf{N}}_2 = \vec{\mathbf{n}}_1 - \vec{\mathbf{n}}_3.$$

We have to choose the boundary curves such that the compatibility condition (27) is satisfied. The construction of each boundary, however, is to depend solely on the data at its end vertices. In order to satisfy the above compatibility conditions we use singular points at the vertices, i.e. $\mathbf{x}_1(0,0,1) = \mathbf{x}_2(0,0,1) = \vec{\mathbf{0}}$. The use of singular points is quite popular for constructing surface splines, see [16] and the references cited therein. In our case, the singular point at $\mathbf{v}_3 = \mathbf{x}(0,0,1)$ entails the coinciding control points

$$\mathbf{v}_3 = \mathbf{p}_{0,0,n} = \mathbf{p}_{1,0,n-1} = \mathbf{p}_{0,1,n-1}. \tag{28}$$

As we want to obtain a surface patch with a well-defined tangent plane at $\mathbf{x}(0,0,1)$, despite the singular parameterization, the control points

$$\mathbf{p}_{0,0,n},\ \mathbf{p}_{2,0,n-2},\ \mathbf{p}_{1,1,n-2},\ \text{and } \mathbf{p}_{0,2,n-2} \tag{29}$$

have to be contained within one plane. For LN surface patches with singular vertices, this compatibility condition is automatically satisfied, as these patches fulfill the condition (15).

4.2 Compatibility along the boundaries

In order to derive another compatibility condition, we consider the LN surface patch (9) in the monomial representation which is obtained after eliminating the barycentric parameter $w = 1 - u - v$,

$$\begin{aligned} \mathbf{y}(u,v) &= \mathbf{x}(u,v,1-u-v) \\ &= \mathbf{a}_n\, u^n + \mathbf{a}_{n-1}\, u^{n-1} + \ldots + \mathbf{a}_1\, u + \mathbf{a}_0 \\ &+ (\mathbf{b}_{n-1}\, u^{n-1} + \mathbf{b}_{n-2}\, u^{n-2} + \ldots + \mathbf{b}_1\, u + \mathbf{b}_0)\, v \\ &+ (\ \ldots\ )\, v^2 + \ldots . \end{aligned} \tag{30}$$

The coefficients $\mathbf{a}_i,\ \mathbf{b}_j,\ \ldots, \in \mathbb{R}^3$ are certain linear combinations of the control points $\mathbf{p}_{i,j,k}$. Similarly we obtain for the normal vector field

$$\vec{\mathbf{M}}(u,v) = \vec{\mathbf{N}}(u,v,1-u-v) = \vec{\mathbf{m}}_0 + \vec{\mathbf{m}}_1\, u + \vec{\mathbf{m}}_2\, v. \tag{31}$$

The coefficients $\vec{\mathbf{m}}_i \in \mathbb{R}^3$, are certain linear combinations of the given normal vectors $\vec{\mathbf{n}}_i$.

We assume that the boundary curve $v = 0$, i.e. $\mathbf{x}(t, 0, 1-t)$, of the LN surface patch has been constructed somehow. Thus, the coefficients $\mathbf{a}_0, \ldots, \mathbf{a}_n$ are already known; they satisfy the LN condition

$$\left(\frac{\partial}{\partial u}\mathbf{y}, \vec{\mathbf{M}} \right)\Big|_{(u,v)=(t,0)} \equiv (\, n\, \mathbf{a}_n\, t^{n-1} + \ldots + \mathbf{a}_1,\ \vec{\mathbf{m}}_0 + \vec{\mathbf{m}}_1\, t\,) \equiv 0, \tag{32}$$

see (15). Now we want to find a LN surface patch with this boundary curve. Of course, the patch has to satisfy the remaining orthogonality condition (15) along the boundary,

$$\left(\frac{\partial}{\partial v}\mathbf{y}, \vec{\mathbf{M}} \right)\Big|_{(u,v)=(t,0)} \equiv \big(\mathbf{b}_{n-1}\, t^{n-1} + \ldots + \mathbf{b}_0,\ \vec{\mathbf{m}}_0 + \vec{\mathbf{m}}_1\, t \big) \equiv 0. \tag{33}$$

In particular, by comparing the coefficients of t^n we obtain the condition $(\mathbf{b}_{n-1}, \vec{\mathbf{m}}_1) = 0$. On the other hand, differentiating the LN condition $(\frac{\partial}{\partial u}\mathbf{y}, \vec{\mathbf{M}}) = 0$ with respect to v gives

$$\left(\frac{\partial}{\partial u}\frac{\partial}{\partial v}\mathbf{y}, \vec{\mathbf{M}} \right)\Big|_{(u,v)=(t,0)} + \left(\frac{\partial}{\partial u}\mathbf{y}, \frac{\partial}{\partial v}\vec{\mathbf{M}} \right)\Big|_{(u,v)=(t,0)} = 0, \tag{34}$$

hence

$$\begin{aligned} &\big((n-1)\, \mathbf{b}_{n-1}\, t^{n-2} + \ldots + \mathbf{b}_1,\ \vec{\mathbf{m}}_1\, t + \vec{\mathbf{m}}_0 \big) \\ &\qquad + \big(n\, \mathbf{a}_n\, t^{n-1} + \ldots \mathbf{a}_1,\ \vec{\mathbf{m}}_2 \big) \equiv 0. \end{aligned} \tag{35}$$

Comparing the coefficients of t^{n-1} we arrive at the compatibility condition

$$(n-1)\ \underbrace{(\mathbf{b}_{n-1},\ \vec{\mathbf{m}}_1)}_{=0} + n\ (\mathbf{a}_n,\ \vec{\mathbf{m}}_2) = 0. \tag{36}$$

Thus, the leading coefficient $\mathbf{a}_n$ of the boundary curve has to satisfy $(\mathbf{a}_n,\ \vec{\mathbf{m}}_2) = 0$. In order to fulfill this condition automatically, we choose the boundary curves of a LN surface patch of degree n as Bézier curves of degree $n-1$, i.e. we set $\mathbf{a}_n = \mathbf{0}$.

4.3 Construction of the boundary curves

Consider the boundary curve joining the vertices $\mathbf{v}_1$ and $\mathbf{v}_2$. It is constructed as a quintic Bézier curve with double boundary control points,

$$\mathbf{x}(1-t,t,0) = \mathbf{v}_1(B_0^5(t)+B_1^5(t)) + \mathbf{b}_2 B_2^5(t) + \mathbf{b}_3 B_3^5(t) + \mathbf{v}_2(B_4^5(t)+B_5^5(t)),$$

with the univariate Bernstein polynomials $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$. By choosing double control points at the boundaries we satisfy the first compatibility condition. The inner control points $\mathbf{b}_2$, $\mathbf{b}_3 \in \mathbb{R}^3$ are as yet unknown.

Owing to (15), the boundary curve has to satisfy the LN condition

$$\left(\frac{\partial}{\partial t}\mathbf{x}(1-t,t,0),\ \vec{\mathbf{N}}(1-t,t,0) \right) \equiv 0, \tag{37}$$

or, equivalently,

$$\Big((\mathbf{b}_2-\mathbf{v}_1)\, B_1^4(t) + (\mathbf{b}_3-\mathbf{b}_2)\, B_2^4(t) + (\mathbf{v}_2-\mathbf{b}_3)\, B_3^4(t),\ (1-t)\,\vec{\mathbf{n}}_1 + t\,\vec{\mathbf{n}}_2 \Big) \equiv 0. \tag{38}$$

By comparing the coefficients of the curve parameter t, we obtain a system of 4 linear equations for the 6 unknown components of the middle control points $\mathbf{b}_2$ and $\mathbf{b}_3$. Among the solutions of this system, we choose the one which minimizes the quadratic objective function

$$F(\mathbf{b}_2,\mathbf{b}_3) = \|-\mathbf{v}_2+\mathbf{b}_3\|^2 + \|\mathbf{v}_2 - 2\,\mathbf{b}_3 + \mathbf{b}_2\|^2 + \|\mathbf{b}_3 - 2\,\mathbf{b}_2 + \mathbf{v}_1\|^2 + \|\mathbf{b}_2 - \mathbf{v}_1\|^2.$$

This function measures the sum of the squared lengths of the control polygon for the second derivative vector of the boundary curve. Clearly, one might also use more sophisticated functionals in order to find the boundaries. Such functionals have been introduced for constructing so–called minimum–norm–network of boundary curves connecting given data, see [13, 14].

The solution $\mathbf{b}_2$, $\mathbf{b}_3$ to the constrained quadratic optimization problem $F(\mathbf{b}_2,\mathbf{b}_3) \to$ Min subject to (38) can be computed with the help of Lagrangian multipliers. This leads to a system of linear equations for the middle control points. Using computer algebra tools it is even possible to find explicit formulas for the inner control points in terms of the given data. A unique solution can be shown to exist, provided that the given normal vectors $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are linearly independent.

Analogously we compute the control points along the remaining two boundaries, joining the vertices $\mathbf{v}_1$ (resp. $\mathbf{v}_2$) with $\mathbf{v}_3$.

4.4 Filling–in the patches

We compute a LN surface patch of degree $n = 6$ which matches the given vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and the normal field (14). Firstly we raise the degree of the Bernstein–Bézier representation of the three boundaries from 5 to 6. This leads to the control points $(\mathbf{p}_{i,j,k})_{i+j+k=6}$ along the boundaries. Owing to the construction of the boundaries and to the degree elevation, both compatibility conditions are automatically satisfied. The first one (27) is fulfilled as the first derivative vectors at the three vertices vanish. The second one (36) is satisfied, because the boundary curves of the sextic patch are chosen as quintic curves.

We compute the inner control points $\mathbf{p}_{i,j,k}$ by solving a system of linear equations. From (16) we obtain certain linear equations that guarantee the property (15) of the LN patch. After eliminating dependencies we obtain a system of 29 equations for 30 unknowns (the components of the inner control points). Among its solutions we choose the one which minimizes the quadratic objective function

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k=n-2}} \left((\Delta_1)^2 + (\Delta_2)^2 + (\Delta_3)^2 \right) \mathbf{p}_{i,j,k}. \tag{39}$$

Similar to the construction of the boundary curves, this objective function is the sum of the squared lengths of the control polygons for the second directional derivative vectors, cf. (10). In this case, however, we cannot recommend using Lagrangian multipliers, as this technique would almost double the size of the system of equations. It is more appropriate to compute the set solution of the system (16) firstly, and then to pick the particular solution that minimizes the objective function.

4.5 An example

Two examples of surfaces that can be obtained from the above construction have already been presented in Figure 1. Here we show another example which demonstrates that the construction is suitable for building surface splines. Two adjacent LN surface patches have been constructed by interpolating four data points with associated normal vectors (shown by the thick grey lines). Along the common boundary, the patches meet with G^1 continuity, since both the boundary curve and the normal vectors are identical.

As observed in our numerical experiments so far, the normal vector field has to be adjusted carefully in order to obtain regular surface patches. Note that the LN surface patch also depends on the scaling of the given normals $\vec{\mathbf{n}}_i$, not only on their directions! We usually obtained good results for data taken from an underlying convex surface. For this class of surfaces, there seems to be a good chance to find an approximating LN surface spline by applying the above construction to a triangulated set of sample points, simply by increasing the number and the density of the data.

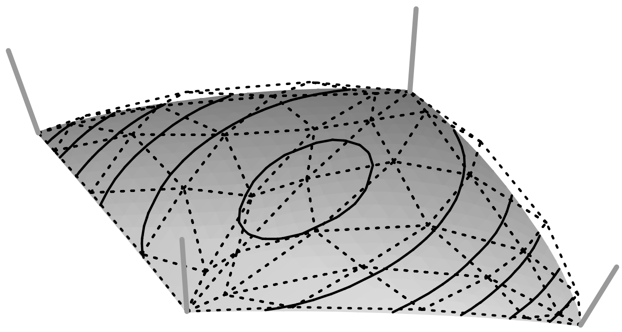

Figure 2: Two triangular LN surface patches joined with G^1 continuity along the common boundary.

5 Concluding remarks

With the help of the ideas presented in this paper it is possible to construct integral surface patches with rational offset surfaces. The possibilities offered by the class of LN surfaces have been demonstrated by presenting a construction of triangular patches from Hermite boundary data which is suitable for building surface splines.

Forthcoming research will mainly focus on conditions that guarantee regular solutions. Currently we are working on a surface spline scheme without singular points at the vertices. Also, we will try to study the asymptotic behaviour (when increasing the number and the density of the input data) of the surface splines.

A similar approach as described in this paper can be applied to integral Bézier surface patches with a Pythagorean field of normal vectors. Again, the property of the Pythagorean normals can be used as an additional linear constraint onto the linear space of integral patches. As an advantage, the offsets of such patches would immediately be rational, without requiring a reparameterization. The systems of constraint equations (cf. (16)), however, would be far bigger, as the degree of the polynomials (15) is higher. Moreover, the construction of a suitable field of Pythagorean normal vectors is much more difficult, as the simplest representatives (which correspond to quadratic triangular or bi-quadratic tensor–product patches on the unit sphere) satisfy certain compatibility conditions of the boundaries, see [5]. In order to avoid these conditions, one would need to use either quartic triangular patches or tensor–product patches of degree (2,4). For these reasons we have decided to study surfaces with a linear field of normal vectors instead.

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