Using Line Congruences for Parameterizing Special Algebraic Surfaces

Dedicated to the memory of Professor Dr. Josef Hoschek

Bert Jüttler and Katharina Rittenschober

Johannes Kepler University, Dept. of Applied Geometry,
Altenberger Str. 69, 4040 Linz,
bert.juettler@jku.at,
URL: http://www.ag.jku.at

Abstract. Surfaces in line space are called line congruences. We consider several special line congruences forming a fibration of the three–dimensional space. These line congruences correspond to certain special algebraic surfaces. Using rational mappings associated with the line congruences, it is possible to generate rational curves and surfaces on them. This approach is demonstrated for quadric surfaces, cubic ruled surfaces, and for Veronese surfaces and their images in three–dimensional space (quadratic triangular Bézier surfaces).

1 Introduction

Line Geometry – the geometry of lines in three–dimensional space – is a classical part of geometry, whose origins can be traced back to works of Plücker in the 19th century. Differential line geometry studies line manifolds using the techniques provided by differential geometry [11, 12]. Recently, computational line geometry [17] has been demonstrated to be useful for various branches of applied geometry, ranging from robot kinematics to computer aided geometric design.

Two–dimensional manifolds of lines are called line *congruences*. A simple example is the system of normals of a surface. Via the Klein mapping, which identifies each line with a point on a hyperquadric in a five–dimensional real projective space, line congruences correspond to surfaces. We are mainly interested in special line congruences, which are equipped with an associated rational mapping.

Linear congruences are the simplest class of line congruences. They have been used for parameterizing the various types of quadric surfaces [8, 6, 16]. The parameterization is based on the quadratic mapping which is associated with them.

After summarizing this approach from the viewpoint of line geometry, we generalize it to other classes of algebraic surfaces. By using other, more sophisticated line congruences, we derive similar results for the various types of cubic ruled surfaces, and for Veronese surfaces.

The paper is organized as follows. First we summarize some fundamental concepts from line geometry. Section 3 discusses line models of quadric surfaces, which are related to the generalized stereographic projection. Section 4 is devoted to cubic ruled surfaces, which are shown to be closely connected with a certain class of line congruences. Similarly, section 5 deals with Veronese surfaces. Finally, we conclude this paper.

2 Line geometry

In this section we summarize the fundamentals of the geometry of lines in three–dimensional space. Due to space limitations, this section can give only an outline of this fascinating branch of geometry. For further information, the reader should consult suitable textbooks, such as [11, 12, 17].

2.1 Homogeneous coordinates

Throughout this paper, points in three–dimensional space will be described by *homogeneous coordinate vectors*

$$\mathbf{p} = (p_0, p_1, p_2, p_3)^\top \in \mathbb{R}^3 \setminus \{(0,0,0,0)^\top\}. \tag{1}$$

Any two linearly dependent vectors correspond to the same point. The associated Cartesian vectors of points satisfying $p_0 \neq 0$ are

$$\underline{\mathbf{p}} = (\frac{p_1}{p_0}, \frac{p_2}{p_0}, \frac{p_3}{p_0})^\top. \tag{2}$$

Points with $p_0 = 0$ are called ideal points or *points at infinity;* they can be identified with the ∞^2 equivalence classes of parallel lines in three–dimensional space.

2.2 Plücker's line coordinates

The line $\mathcal{L}$ spanned by two different points $\mathbf{p}$, $\mathbf{q}$ (i.e., with linearly independent homogeneous coordinate vectors) consists of all points

$$\mathbf{x} = \lambda\mathbf{p} + \mu\mathbf{q}, \quad (\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3}$$

Consider the 2×2 determinants $l_{ij} = p_i\, q_j - p_j\, q_i$. They produce essentially six different numbers

$$\mathbf{L} = \mathbf{p} \wedge \mathbf{q} = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})^\top. \tag{4}$$

The components of the vector $\mathbf{L} \in \mathbb{R}^6$, which are called the *Plücker coordinates*, are homogeneous coordinates of the line $\mathcal{L}$. They do not depend on the choice of the points $\mathbf{p}$ and $\mathbf{q}$. Indeed, the two points $\mathbf{p}' = \lambda_0\mathbf{p} + \mu_0\mathbf{q}$ and $\mathbf{q}' = \lambda_1\mathbf{p} + \mu_1\mathbf{q}$ lead to the modified Plücker coordinates $\mathbf{L}'$ which are linearly dependent on $\mathbf{L}$,

$$\mathbf{L}' = \mathbf{p}' \wedge \mathbf{q}' = \det \begin{pmatrix} \lambda_0 & \lambda_1 \\ \mu_0 & \mu_1 \end{pmatrix} \mathbf{L}. \tag{5}$$

The Plücker coordinates l_{ij} of a line satisfy the *Plücker's identity*

$$l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0. \tag{6}$$

If $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, $\mathbf{s}$ are two pairs of points spanning two lines $\mathbf{L}$ and $\mathbf{M}$, respectively, then the determinant of the 4×4 matrix $[\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}]$ can be expanded to

$$\langle \mathbf{L}, \mathbf{M} \rangle = l_{01}m_{23} + l_{02}m_{31} + l_{03}m_{12} + m_{01}l_{23} + m_{02}l_{31} + m_{03}l_{12} = 0. \tag{7}$$

Consequently, the two lines intersect if and only if $\langle \mathbf{L}, \mathbf{M} \rangle = 0$. Plücker's identity is obtained the special case $\mathbf{L} = \mathbf{M}$, i.e., $\frac{1}{2}\langle \mathbf{L}, \mathbf{L} \rangle = 0$.

Remark 1. If $\mathbf{q} = (0, v_1, v_2, v_3)$ is chosen as an ideal point, and $\mathbf{p} = (1, \underline{q}_1, \underline{q}_2, \underline{q}_3)$ is the vector of Cartesian coordinates, homogenized by adding a leading 1, then the Plücker coordinates

$$\mathbf{L} = (\underbrace{v_1, v_2, v_3}_{\text{direction vector}}, \underbrace{(v_1, v_2, v_3) \times (\underline{q}_1, \underline{q}_2, \underline{q}_3)}_{\text{momentum vector}})^\top. \tag{8}$$

are the so–called momentum vector (which is perpendicular to the plane spanned by the line, and whose length is equal to the distance from the origin) and the direction vector of the line, see Figure 1. Plücker's identity (6) is satisfied, since direction and momentum vector are mutually perpendicular.

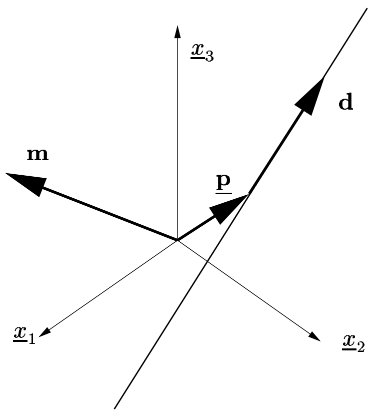

Fig. 1. Momentum vector $\mathbf{m}$ and direction vector $\mathbf{d}$ of a line $\mathcal{L}$

Remark 2. An alternative notation is based on three–dimensional vectors from $\mathbb{R}^3 + \varepsilon\mathbb{R}^3$, where the so–called *"dual unit"* ε satisfies $\varepsilon^2 = 0$. Dual unit vectors $\mathbf{u} = \mathbf{d} + \varepsilon\mathbf{m}$ satisfying $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{d}^2 + \varepsilon\mathbf{d}, \mathbf{m}$ correspond to the line with direction vector $\mathbf{d}$ and the momentum vector $\mathbf{m}$. Two lines intersect if and only if the dual part of the inner product vanishes. The inner product can be used to determine both the distance and the angle between two lines. This notation is frequently used in space kinematics [2], where it leads to the dual quaternion representation of rigid body motions.

2.3 Klein's mapping

Using Plücker coordinates, any line $\mathcal{L}$ in three–dimensional space is identified with the point

$$\mathbf{L} = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})^\top \tag{9}$$

in the five–dimensional real projective space $P^5(\mathbb{R})$, which is contained in the hyperquadric M given by Plücker's identity (6). On the other hand, any such point $\mathbf{L}$ corresponds to a unique line. Points with $l_{01} = l_{02} = l_{03} = 0$ correspond to points at infinity. This bijective mapping

$$\text{line } \mathcal{L} \subset P^3(\mathbb{R}) \quad \mapsto \quad \text{point } \mathbf{L} \in M \subset P^5(\mathbb{R}) \tag{10}$$

is called the *Klein mapping*. The point $\mathbf{L}$ is called the *Klein image* of the line $\mathcal{L}$.

The polar hyperplane of a point $\mathbf{L}$ with respect to the hyperquadric M,

$$\Pi_{\mathbf{L}} = \{ \mathbf{X} \mid \langle \mathbf{X}, \mathbf{L} \rangle = 0 \}, \tag{11}$$

intersects the hyperquadric M in the Klein images of all lines which intersect the given line $\mathcal{L}$.

Remark 3. The homogeneous coordinates of points in five–dimensional real projective space $P^5(\mathbb{R})$ will also be indexed as usual,

$$\mathbf{P} = (p_0, p_1, p_2, p_3, p_4, p_5)^\top. \tag{12}$$

If such a point corresponds to the Plücker coordinates $\mathbf{L}$ of a line $\mathcal{L}$, then the coordinates are identified according to $p_0 = p_{01}$, $p_1 = p_{02}$, $p_2 = p_{03}$, $p_3 = p_{23}$, $p_4 = p_{31}$, $p_5 = p_{12}$.

2.4 Curves in line space

Any *curve* $\mathbf{L}(u)$, $u \in (a, b) \subseteq \mathbb{R}$, which is fully contained in the hyperquadric M, is the Klein image of a one–parameter family of lines, i.e., of a *ruled surface.* The Klein images of the generators are the points of the curve.

As an example, we consider all lines which intersect three given lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$. They form a so–called *regulus*, which is one of the two systems of straight lines on a ruled quadric surface.

The Klein image of a regulus is the intersection of the three polar hyperplanes $\Pi_{\mathbf{L}_1}$, $\Pi_{\mathbf{L}_2}$, $\Pi_{\mathbf{L}_3}$ of the three points $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ with the hyperquadric M. The three polar hyperplanes intersect in a two–dimensional plane. Consequently, the Klein image of the regulus is simply a conic on the hyperquadric M.

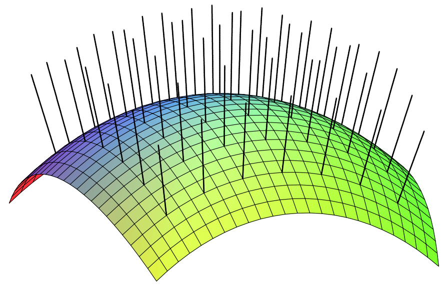

Fig. 2. Normal congruence of a surface.

2.5 Surfaces in line space

Now consider a (two-dimensional) surface $\mathbf{L}(u, v)$ which is fully contained in the hyperquadric M. It is the Klein image of a two–parameter family of lines. Such a system of lines is called a *line congruence.*

As an example we consider the *normal congruence* of a surface $\underline{\mathbf{x}}(u, v)$, $(u, v) \in \Omega \subset \mathbb{R}^1$, which consists of all normals

$$\mathbf{x}(u, v) + \lambda \mathbf{n}(u, v), \tag{13}$$

where $\mathbf{n}(u, v)$ is the field of the normal vectors of the given surface, see Figure 2. The Klein image is the surface

$$\mathbf{L}(u, v) = (\ \mathbf{n}(u, v)^\top\ ,\ [\mathbf{n}(u, v) \times \underline{\mathbf{x}}(u, v)]^\top\ )^\top. \tag{14}$$

Normal congruences of surfaces have been used in order to detect the shortest distance between free–form surfaces [24]. In line space, this task can be formulated as a problem of surface–surface intersection.

A line congruence is said to have the *space–filling property*, if any point is contained in exactly one line, except for the points on finitely many curves. With other words, the line congruence forms a *fibration* of the three–dimensional space. In this situation, the exceptional curve(s) will be called the *focal curve(s)* of the line congruence.

Such congruences can be used for defining rational mappings on algebraic surfaces. Several examples will be discussed in the remainder of this paper.

Remark 4. 1. Space filling line congruences without exceptions (i.e., without focal curves) are called *spreads*; they have been analyzed in the field of Foundations of Geometry (see e.g. [18]). Line congruences and spreads are also of recent interest in Computer Vision.

2. Using a notion from the classical theory of algebraic line geometry, space filling line congruences are characterized by having the *bundle degree* 1 – the number of lines passing through a generic point equals one.

3 Line models of quadric surfaces

Consider two skew lines $\mathcal{F}_1$ and $\mathcal{F}_2$ in three–dimensional space. For any point $\mathbf{p}$, which does not belong to one of these lines, the two planes spanned by $\mathbf{p}$ and either line intersect in a unique line $\mathcal{L}(\mathbf{p})$. Clearly $\mathcal{L}(\mathbf{p})$ passes through $\mathbf{p}$ and intersects both $\mathcal{F}_1$ and $\mathcal{F}_2$.

The two–parameter family of lines obtained in this way is called a *linear congruence of lines.* It consists of all lines connecting arbitrary points on the lines $\mathcal{F}_1$ and $\mathcal{F}_2$. Clearly, this line congruence has the space–filling property with the two focal lines $\mathcal{F}_1$ and $\mathcal{F}_2$.

The Klein image of the line congruence is the intersection of the Klein quadric M with the two polar hyperplanes $\Pi_{\mathbf{F}_1}$ and $\Pi_{\mathbf{F}_2}$. Since the intersection of two hyperplanes in a five–dimensional space is three–dimensional, we obtain a quadric surface in a three–dimensional space. Depending on the choice of the focal lines, we get the two different types of non–singular quadric surfaces: ruled quadrics, which are projectively equivalent to the hyperboloid of revolution, and oval quadrics, which are projectively equivalent to a sphere. As observed in [16], this leads to an alternative approach to the so–called *generalized stereographic projection* [8, 9], which has been shown to be a useful tool for generating rational curves and surfaces on oval and ruled quadric surfaces.

3.1 Ruled quadrics

We consider the two real focal lines

$$\begin{array}{l}\mathcal{F}_1 = \{\mathbf{p} \mid \mathbf{p} = (0,0,\lambda,\mu)^\top, \lambda,\mu \in \mathbb{R}\} \quad \text{and} \\ \mathcal{F}_2 = \{\mathbf{p} \mid \mathbf{p} = (\lambda,\mu,0,0)^\top, \lambda,\mu \in \mathbb{R}\}.\end{array} \tag{15}$$

These lines are the infinite line which is shared by all planes parallel to the $(\underline{x}_2, \underline{x}_3)$–plane, and the $\underline{x}_1$–axis, respectively. The resulting line congruence $\widehat{\mathcal{R}}$ is shown in Figure 3 (top right). The Plücker coordinates of the focal lines are

$$\mathbf{F}_1 = (0,0,0,1,0,0)^\top \quad \text{and} \quad \mathbf{F}_2 = (1,0,0,0,0,0)^\top \tag{16}$$

Hence, due to the intersection condition (7), the Klein image of the congruence satisfies the two linear equations $\langle \mathbf{L}, \mathbf{F}_1 \rangle = \langle \mathbf{L}, \mathbf{F}_2 \rangle = 0$, or, equivalently, $l_{01} = l_{23} = 0$, and Plücker's identity simplifies to

$$l_{02} l_{31} + l_{03} l_{12} = 0. \tag{17}$$

This is the equation of a *ruled quadric surface* $\mathcal{R}$ in the three–dimensional subspace of $P^5(\mathbb{R})$, which is given by $l_{01} = l_{23} = 0$.

For any point $\mathbf{p} = (p_0, p_1, p_2, p_3)^\top$, the plane spanned by $\mathbf{p}$ and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) intersects $\mathcal{F}_2$ (resp. $\mathcal{F}_1$) in the point $\mathbf{p}^* = (p_0, p_1, 0, 0)^\top$ (resp. $\mathbf{p}_* = (0, 0, p_1, p_2)^\top$). Consequently, the Plücker coordinates of the line of the congruence through $\mathbf{p}$ are

$$\mathbf{L}_{\widehat{\mathcal{R}}}(\mathbf{p}) = \mathbf{p}^* \wedge \mathbf{p}_* = (0, p_0 p_2, p_0 p_3, 0, -p_3 p_1, p_1 p_2)^\top \tag{18}$$

The mapping $\mathbf{p} \mapsto \mathbf{L}_{\widehat{\mathcal{R}}}(\mathbf{p})$ is equivalent to the *generalized stereographic projection* onto the hyperbolic paraboloid, as introduced in [8, 9].

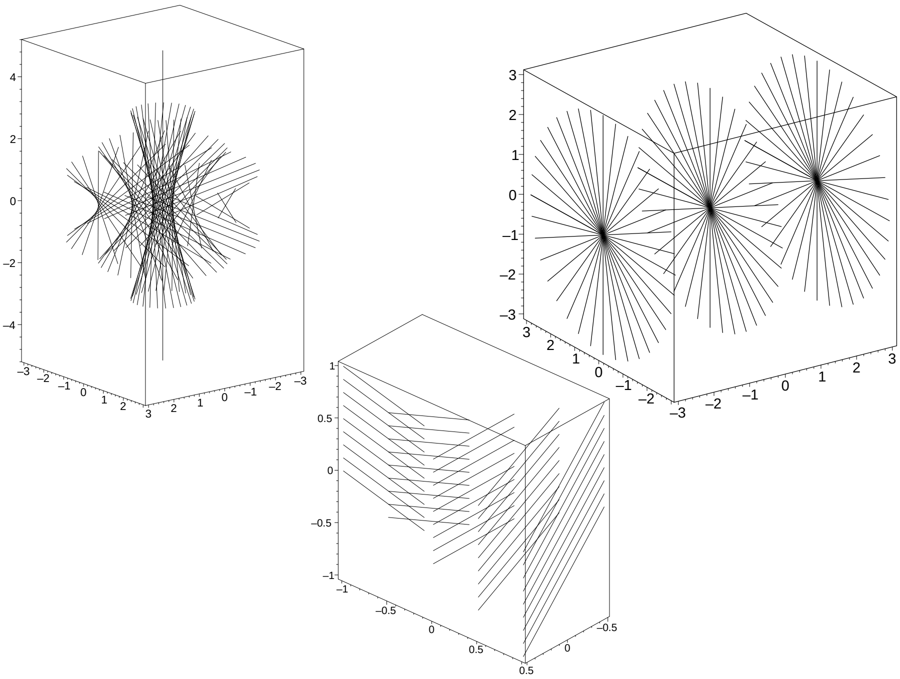

Fig. 3. Elliptic, hyperbolic and parabolic linear congruence.

3.2 Oval quadrics

In the complex extension of the real projective 3–space, we consider the two conjugate–complex focal lines

$$\begin{aligned}\mathcal{F}_1 &= \{\mathbf{p} \mid \mathbf{p} = (\lambda, \mu, -i\mu, i\lambda)^\top, \lambda, \mu \in \mathbb{C}\} \quad \text{and} \\ \mathcal{F}_2 &= \{\mathbf{p} \mid \mathbf{p} = (\lambda, \mu, i\mu, -i\lambda)^\top, \lambda, \mu \in \mathbb{C}\}.\end{aligned} \tag{19}$$

The resulting line congruence $\widehat{\mathcal{O}}$ is shown in Figure 3 (top left). The Plücker coordinates of the focal lines are

$$\mathbf{F}_1 = (1, -i, 0, -1, i, 0)^\top \quad \text{and} \quad \mathbf{F}_2 = (1, i, 0, -1, -i, 0)^\top. \tag{20}$$

Hence, again due to the intersection condition (7), the Klein image of the congruence satisfies the two linear equations $\langle \mathbf{L}, \mathbf{F}_1 + \mathbf{F}_2 \rangle = \langle \mathbf{L}, \mathbf{F}_1 - \mathbf{F}_2 \rangle = 0$, which lead to the two conditions $l_{01} = l_{23}$ and $l_{02} = l_{31}$. Consequently, Plücker's identity simplifies to

$$l_{01}^2 + l_{02}^2 + l_{03} l_{12} = 0. \tag{21}$$

This is the equation of a *oval quadric surface* $\mathcal{O}$ in the three–dimensional subspace of $P^5(\mathbb{R})$, which is given by $l_{01} = l_{23}$ and $l_{02} = l_{31}$. In fact, by introducing Cartesian coordinates according to

$$1 : X : Y : Z = (-l_{12}) : l_{01} : l_{02} : l_{03}, \tag{22}$$

equation (21) is transformed into the elliptic paraboloid $Z = X^2 + Y^2$.

For any real point $\mathbf{p} = (p_0, p_1, p_2, p_3)^\top$, the plane spanned by $\mathbf{p}$ and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) intersects $\mathcal{F}_2$ (resp. $\mathcal{F}_1$) the point

$$\begin{aligned} \mathbf{p}^+ &= (p_0 + ip_3, p_1 - ip_2, p_2 + ip_1, p_3 - ip_0)^\top \\ (\text{resp. } \mathbf{p}_+ &= (p_0 - ip_3, p_1 + ip_2, p_2 - ip_1, p_3 + ip_0)^\top\ ). \end{aligned} \tag{23}$$

These two intersections are conjugate complex. A real point on the congruence line through the point $\mathbf{p}$ can be generated by taking the linear combination

$$\mathbf{p}^\perp = \frac{i}{2}(\mathbf{p}_+ - \mathbf{p}^+) = (-p_3, p_2, -p_1, p_0)^\top . \tag{24}$$

Consequently, the Plücker coordinates of the line of the congruence through $\mathbf{p}$ are

$$\begin{aligned} \mathbf{L}_{\widehat{\mathcal{O}}}(\mathbf{p}) &= \mathbf{p} \wedge \mathbf{p}^\perp \\ &= (p_0p_2{+}p_1p_3, p_2p_3{-}p_0p_1, p_0^2{+}p_3^2, p_0p_2{+}p_1p_3, p_2p_3{-}p_0p_1, -p_1^2{-}p_2^2)^\top \end{aligned} \tag{25}$$

The mapping $\mathbf{p} \mapsto \mathbf{L}_{\widehat{\mathcal{O}}}(\mathbf{p})$ is equivalent to the *generalized stereographic projection* onto the unit sphere, as introduced in [8, 9].

3.3 Images of lines

Consider a line $\mathcal{C}$ in 3–space, which does not intersect both focal lines. All lines of the congruence which pass through $\mathcal{C}$ form a regulus of lines. The Klein image of this regulus is a conic section on the quadric. Consequently, the images of the points of $\mathcal{C}$ under the generalized stereographic projections $L_{\widehat{\mathcal{R}}}$, $L_{\widehat{\mathcal{O}}}$ form conics.

If the line $\mathcal{C}$ intersects one of the focal lines, say $\mathcal{F}_1$, then the regulus degenerates into a pencil of lines, since all its lines pass through the intersection of the plane spanned by $\mathcal{C}$ and $\mathcal{F}_1$ with the second focal line $\mathcal{F}_2$. In this case, the images of the points of $\mathcal{C}$ under the generalized stereographic projection (18) form a line. Clearly, only real focal lines lead to real lines on the quadric, hence hyperbolic (resp. elliptic) linear congruences correspond to ruled (resp. oval) quadric surfaces.

The Klein image of any line $\mathcal{L}$ of the linear congruence is a point $\mathbf{L}$ on the quadric surface. Those lines of the congruence, which are contained in the two planes spanned by the line $\mathcal{L}$ and either one of the two focal lines are mapped to the two generating lines of the quadric $\mathcal{R}$ through that point. In the ruled quadric case, these two lines are real, otherwise they are conjugate complex.

3.4 Cones and cylinders

The Klein images $\mathbf{F}_1$, $\mathbf{F}_2$ of the two focal lines span a line in $P^5(\mathbb{R})$ which intersects the hyperquadric M in two real points (hyperbolic case) or in two conjugate-complex points (elliptic case). The Klein image of the line congruence is the intersection of the polar 4–plane of this line with the hyperquadric M.

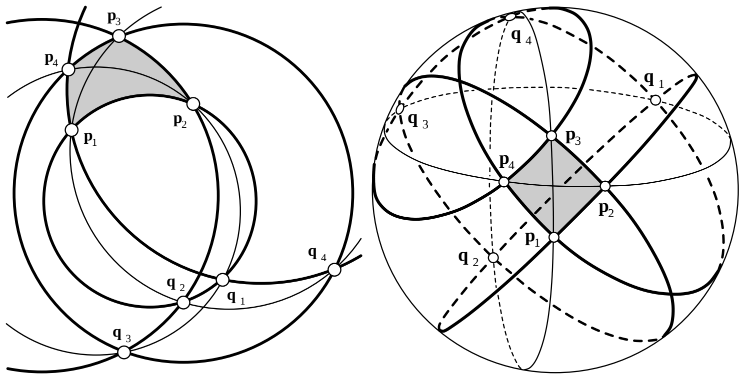

Fig. 4. Miquel's theorem in the plane and on the sphere: The circle through $\mathbf{p}_1, \mathbf{p}_3, \mathbf{q}_2, \mathbf{q}_4$ exists if and only if the circle through $\mathbf{p}_2, \mathbf{p}_4, \mathbf{q}_1, \mathbf{q}_3$ exists too. Four spherical arcs can act as the boundaries of a biquadratic patch iff both additional circles exist.

If the line in $P^5(\mathbb{R})$ touches the hyperquadric M, then one obtains the Klein image of a *parabolic line congruence* (see Figure 3, bottom), which is a singular quadric (cone or cylinder). Similar to the case of oval and ruled quadrics, an associated generalized stereographic projection can be obtained, see [6] for more information.

3.5 Additional algebraic properties

As shown in [8], the mappings $L_{\widehat{\mathcal{O}}}$ and $L_{\widehat{\mathcal{R}}}$ have a very useful algebraic property: *any irreducible*[1] *rational parametric representation of a curve or surface can be obtained by applying these mappings to an irreducible rational curve or surface.* Without going into detail, we mention two consequences.

Any quadratic triangular Bézier patch on an oval quadric is the image of a linear patch. Consequently, the three boundary curves of the quadratic patch intersect in a single point[2]. This point is the Klein image of the unique line of the congruence, which is contained in the plane spanned by the linear patch.

Any biquadratic tensor–product Bézier patch on an oval quadric is the image of a bilinear patch. Consequently, the four boundary curves of the quadratic patch belong to the configuration of Miquel's theorem, see Figure 4. The boundary curves two additional circles are the images of the edges of the tetrahedron which is spanned by the four control points of the bilinear patch. See [7, 8] for further information and related references.

Note that the generalized stereographic projection onto the unit sphere is also closely related to quaternion calculus and the kinematical mapping of spherical

[1] A rational parametric representation of a curve or surface is said to be irreducible, if the components of the homogeneous coordinates do not share any polynomial factors.

[2] This fact had already been observed in [20].

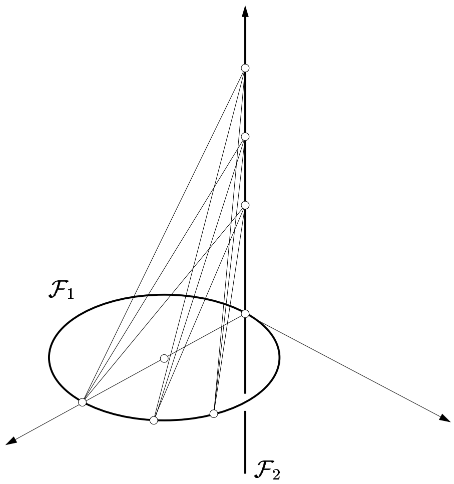

Fig. 5. The line congruence $\widehat{\mathcal{C}}$.

kinematics [17]. Recently, this connection has been exploited for generating spatial Pythagorean hodograph curves [4, 10].

4 Line models of cubic ruled surfaces

We generalize this line–geometric approach to another, more complicated class of line congruences. After analyzing its Klein image, we obtain the various types of cubic ruled surfaces by projecting it back into three–dimensional space.

4.1 The line congruence

The line congruence $\widehat{\mathcal{C}}$ has the two focal curves

$$\begin{aligned}\mathcal{F}_1 &= \{\, \mathbf{p} \mid \mathbf{p} = (\lambda, 0, 0, \mu)^\top,\ \lambda, \mu \in \mathbb{R}\,\}\\ \mathcal{F}_2 &= \{\, \mathbf{p} \mid \mathbf{p} = (p_0, p_1, p_2, 0)^\top,\ p_1^2 - 2p_0p_1 + p_2^2\,\}\\ &= \{\, \mathbf{p} \mid \mathbf{p} = (s^2+t^2, (s+t)^2, s^2-t^2, 0)^\top,\ s, t \in \mathbb{R}\,\}.\end{aligned} \tag{26}$$

The second focal curve is the circle in the plane $\underline{x}_3 = 0$ with radius 1 and center $\underline{\mathbf{c}} = (1, 0, 0)$. The first one is the $\underline{x}_3$–axis.

Clearly, the metric properties of the focal curves are not important. As a projectively equivalent choice one may take any non–degenerate conic section and any line which intersects it, but which is not contained in the same plane.

Lemma 1. *The line congruence $\widehat{\mathcal{C}}$ with the focal curves (26) has the space–filling property: any point $\mathbf{p} = (p_0, p_1, p_2, p_3)^\top$ ($\mathbf{p} \notin \mathcal{F}_1 \cup \mathcal{F}_2$) lies on exactly one line through the two focal curves. This line has the Plücker coordinates*

$$\begin{aligned}\mathbf{L}_{\widehat{\mathcal{C}}} = (&4p_0p_1^2 - 2p_1^3 - 2p_1p_2^2, 4p_0p_1p_2 - 2p_1^2p_2 - 2p_2^3,\\ &-p_3(2p_1^2 + 2p_2^2), -4p_1p_2p_3, 4p_1^2p_3, 0)^\top.\end{aligned} \tag{27}$$

Proof. The plane spanned by $\mathbf{p}$ and the $\underline{x}_3$-axis intersects the circle $\mathcal{F}_2$ in the origin of the Cartesian coordinate system $(1,0,0,0)^\top$ and in the point

$$\mathbf{q}(\mathbf{p}) = (2p_1^2 + 2p_2^2, 4p_1^2, 4p_1p_2, 0)^\top. \tag{28}$$

The Plücker coordinates of the line are $\mathbf{L}_{\widehat{\mathcal{C}}} = \mathbf{p} \wedge \mathbf{q}(\mathbf{p})$. □

Proposition 1. *The Klein image of the line congruence $\widehat{\mathcal{C}}$ is a cubic ruled surface, which is contained in a four–dimensional subspace of $P^5(\mathbb{R})$.*

Proof. The line congruence $\widehat{\mathcal{C}}$ is projectively equivalent to the congruence generated by the two focal curves $(1,0,0,s)^\top$, $s \in \mathbb{R}$ ($\underline{x}_3$–axis), and $(1,t,t^2,0)^2$, $t \in \mathbb{R}$ (a parabola in the plane $\underline{x}_3 = 0$, which will be called the focal parabola). The Plücker coordinates of the lines are

$$\mathbf{L}(s,t) = (t, t^2, -s, -st^2, st, 0)^\top, \quad s,t \in \mathbb{R} \tag{29}$$

The area of the Newton polygon of this surface in $P^5(\mathbb{R})$ equals $3/2$, hence it is a cubic surface (see [14]). On the other hand, it is a ruled surface, since the parameter lines $t =$ constant are lines. □

The Klein images of all lines which pass through a fixed point of the circle $\mathcal{F}_2$ are the generators of the cubic ruled surface. In the framework of projective differential geometry, the cubic ruled surfaces in four–dimensional space have been studied by Weitzenböck and Bos [25].

4.2 Projection back into three–space

The Klein image of the congruence $\widehat{\mathcal{C}}$ is a surface in a four–dimensional subspace of $P^5(\mathbb{R})$. By mapping it back into three–space, we obtain a cubic ruled surface. This mapping is described by a projective transformation

$$\pi : \mathbf{L} \mapsto \pi(\mathbf{L}) = A\ (l_{01}, l_{02}, l_{03}, l_{23}, l_{31})^\top \tag{30}$$

where A is a 4×5 matrix. We assume that A has maximal rank. Otherwise, the image of the surface is contained in a plane. The kernel of A is called the *center* $\mathbf{C}$ of this mapping.

The types of cubic ruled surfaces in three–space. Recall that there are three types of cubic ruled surfaces in three–dimensional space [17], see Figure 6. All are equipped with a unique double line. Each generator intersects the double line. The double line consists of singular points (in the sense of algebraic geometry), and the osculating cone (the zero set of the associated Hessian matrix) degenerates into two (possibly conjugate–complex) planes. For one or two points along the double line, these two planes degenerate into a double plane. These two points are called the cuspidal points of the surface. Depending on their nature, one either obtains a Plücker conoid (two real points), a Zindler conoid (two conjugate–complex points) or Cayley's cubic ruled surface (Cayley surface for short, one cuspidal point).

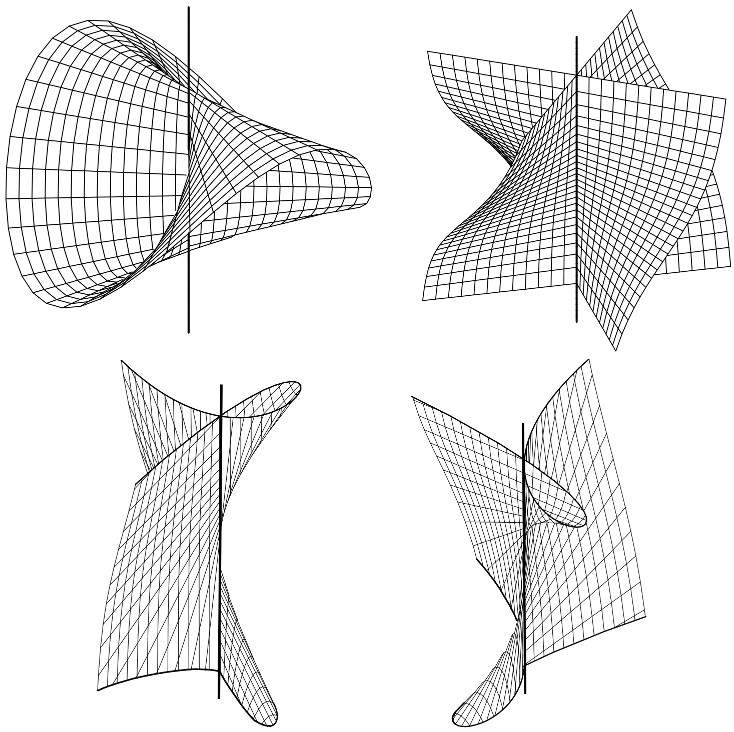

Fig. 6. The three types of cubic ruled surfaces. Top left: Plücker conoid, top right: Zindler conoid, bottom: Cayley surface (two views).

Clearly, the class of cubic ruled surfaces may degenerate in various ways (cubic cones, quadric surfaces, etc.). For the sake of simplicity, we restrict ourself to the generic case. Up to projective mappings, any cubic ruled surface is equivalent to one of the surfaces shown in Figure 6.

Projecting space cubics into planar ones. Before proceeding to four–dimensional space, we discuss a similar situation in three–dimensional space.

Any planar rational cubic curve can be obtained by applying a projective transformation $\pi : P^3(\mathbb{R}) \to P^2(\mathbb{R})$ to the space cubic $\mathbf{c}(t) = (1, t, t^2, t^3)^\top$. The mapping π has a unique center, which is the kernel of the corresponding 3×4–matrix. The location of the center governs the shape of the result. If the center is located on one of the tangents of the space cubic, the image is a planar cubic with a cusp. If the center is even on the curve itself, the image is a conic section (conic for short). Otherwise, the planar cubic has either a double point or an isolated singular point (in the algebraic sense).

Any line connecting two points on the curve is called a chord. A double point is generated by a chord of the curve which passes through the center. A cusp is generated by a tangent of the curve through the center. A isolated singular

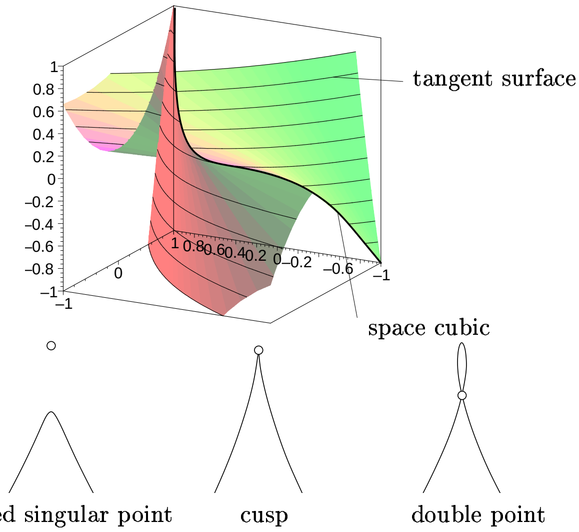

Fig. 7. The three types of planar rational cubics (bottom) can be obtained by projecting a space cubic into the plane. The shape of the result depends on the location of the center of projection with respect to the tangent surface (top) of the cubic. The tangent surface is visualized by level curves.

point is generated by a chord connecting two conjugate–complex points of the curve which passes through the center.

For all centers which are on the same side of the tangent surface (see Figure 7), the type of the singularity is the same[3].

Projecting 4D cubic ruled surfaces into 3D ones. In order to simplify the calculations, we use again the representation (29) of the Klein image of the line congruence which was used in the proof of Proposition 1. This line congruence has a the focal line $(1,0,0,s)^\top$ and the focal parabola $(1,t,t^2,0)^\top$. Moreover, since the Klein image of the line congruence is contained in the hyperplane $l_{12}=0$, we omit the last coordinate throughout the remainder of this paper, i.e.,

$$\mathbf{L}(s,t) = (t, t^2, -s, -st^2, st)^\top, \quad s,t \in \mathbb{R} \tag{31}$$

Consequently, we deal with a 2–surface in real projective 4–space.

Any point of a ruled surface has a 2–dimensional tangent plane. Along each generating line, the union of the tangent planes forms a hyperplane, which will be called the *tangent hyperplane*.

[3] This fact has been exploited for deriving an alternative approach to earlier results [21] on a so–called characterization diagram for planar cubics in Bernstein–Bézier form [16].

Lemma 2. *The one–parameter family of tangent hyperplanes covers an open subset of $P^4(\mathbb{R})$ twice, while the interior of the complementary subset is not covered. The two subsets of $P^4(\mathbb{R})$ are separated by the hyperquadric T,*

$$T = \{\ \mathbf{q} = (q_0, q_1, q_2, q_3, q_4)^\top \mid q_2 q_3 = q_4^2\ \}. \tag{32}$$

Proof. The tangent hyperplanes are spanned by the four points

$$\mathbf{L}\Big|_{s=0,t=t_0}, \quad \frac{\partial}{\partial s}\mathbf{L}\Big|_{s=0,t=t_0}, \quad \frac{\partial}{\partial t}\mathbf{L}\Big|_{s=0,t=t_0}, \quad \frac{\partial}{\partial t}\mathbf{L}\Big|_{s=1,t=t_0}, \tag{33}$$

for any generating line $\mathbf{x}(s, t_0)$, $t_0 \in \mathbb{R}$, constant, $s \in \mathbb{R}$. A short calculation leads to their homogeneous coordinates $(0, 0, t^2, 1, 2t)^\top$. Their envelope can be shown to be the quadric T. □

Remark 5. 1. The situation is similar for the tangents of a conic in the plane: Points in the exterior part of the conic are covered twice, while points in the interior part are not covered, see Figure 8.
2. The hyperquadric T is a degenerate quadric with two–dimensional generating planes. It can be thought of as a cylinder, but with two–dimensional rulings.

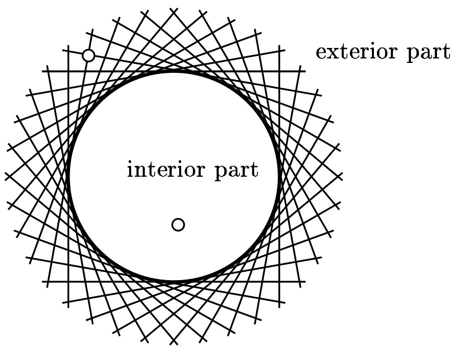

Fig. 8. Tangents of a conic in the plane.

As the next step, we analyze the conic sections on the Klein image of the congruence. These conic section are the Klein images of the reguli which are contained in the congruence.

Lemma 3. *There exists a two–parameter family of conic sections on the Klein image of the congruence. Any conic can be obtained by substituting $s = t/(ct + d)$ in (31), with constants $c, d \in \mathbb{R}$.*

Consider the associated two–parameter family of 2–planes, which are spanned by the conics. Except for the points in the hyperquadrics M and T, each point in four–space belongs to exactly one of those 2–planes.

Proof. Clearly, any curve $\mathbf{L}(t/(ct+d), t)$ is a quadratic rational curve, i.e., a conic section. It remains to be shown that any conic can be obtained in this way.

Any hyperplane intersects the Klein image in a cubic curve. In order to obtain a conic, this cubic has to factor into a conic section and a line. The line has to be one of the generating lines of the Klein image.

The hyperplanes through the generator $\mathbf{L}(s, t_0)$ ($t_0 \in \mathbb{R}$ constant, $s \in \mathbb{R}$) of the Klein image have the homogeneous coordinates

$$\mathbf{H} = (-t_0 k_1,\ k_1,\ t_0(-k_3 t_0 + k_4),\ k_3,\ k_4)^\top, \tag{34}$$

They form a 2–parameter family, since they depend on the three homogeneous parameters $k_1, k_3, k_4 \in \mathbb{R}$.

Indeed, the intersection between hyperplane and surface leads to

$$\mathbf{L}(s,t)^\top \mathbf{H} = -\underbrace{(t_0 - t)}_{\text{generator}}\underbrace{(-s k_3 t_0 + s k_4 + t k_1 - s k_3 t)}_{\text{conic}}. \tag{35}$$

The second factor can be solved for s, which leads to the parametric representations of the conics,

$$s(t) = \frac{t k_1}{(t + t_0) k_3 - k_4} = \frac{t}{ct + d} \quad (k_1 \neq 0). \tag{36}$$

This gives the following representation of the conics on the Klein image of the congruence (which are Klein images of reguli)

$$\mathbf{R}(t) = \mathbf{L}(s(t), t) = ((ct+d),\ (ct+d)t,\ -1,\ -t^2,\ t)^\top, \quad t \in \mathbb{R}. \tag{37}$$

The system of conics depends on 2 parameters $c, d \in \mathbb{R}$.

A point $\mathbf{Q} = (q_0, q_1, q_2, q_3, q_4)$ lies in the same 2–plane as one of these conics if and only if the rank of the 5×4 matrix $(\mathbf{R}(t), \dot{\mathbf{R}}(t), \ddot{\mathbf{R}}(t), \mathbf{Q})$ is less than 4. This condition leads to unique solutions for the parameters c and d,

$$c = \frac{q_0 q_4 + q_1 q_2}{q_4^2 - q_2 q_3}, \quad d = \frac{q_0 q_3 + q_1 q_4}{q_4^2 - q_2 q_3}. \tag{38}$$

Thus, except for the points Q on the quadric surface T which is characterized by the equation $q_4^2 = q_2 q_3$, there is always a unique conic section lying in a 2–plane through it. □

Remark 6. 1. If $q_0 q_3 + q_1 q_4 = 0$, i.e., $d = 0$, then this conic is the Klein image of the cone spanned by the focal parabola and by the point $(1, 0, 0, 1/c)$ on the focal line. This equation characterizes the intersection of the hyperquadric M with the hyperplane $q_5 = q_{12} = 0$.

2. The Lemma can also be concluded from the fact, that a regulus is generated by a projective mapping between a conic and a line intersecting the conic, where the intersection point is a fixed point of the mapping. The mapping is given by the bilinear transformation $t \mapsto s(t)$ (see (36)), which keeps the intersection, due to $s(0) = 0$.

3. The two quadric surfaces M and T intersect in the Klein image $\mathbf{L}(s,t)$ of the line congruence.

Corollary 1. *If the center of the projective mapping $\pi : P^4(\mathbb{R}) \to P^3(\mathbb{R})$ does not belong to the hyperquadric T, then the unique conic on the Klein image $\mathbf{L}(s,t)$, which shares a 2–plane with the center, is mapped to the double line of the cubic ruled surface. The two tangents of the conic through the center touch the conic in two points. These points are mapped to the cuspidal points of the cubic ruled surface. The two points are either real (Plücker conoid) or conjugate complex (Zindler conoid). See Figure 9 for a sketch of the situation in 4–dimensional space.*

Proof. The first part of the corollary is an immediate consequence of the previous Lemma. The second part results from the fact that the two tangents to the conic are also tangents to the Klein image of the line congruence. Consequently, the image surface under π has a singularity. □

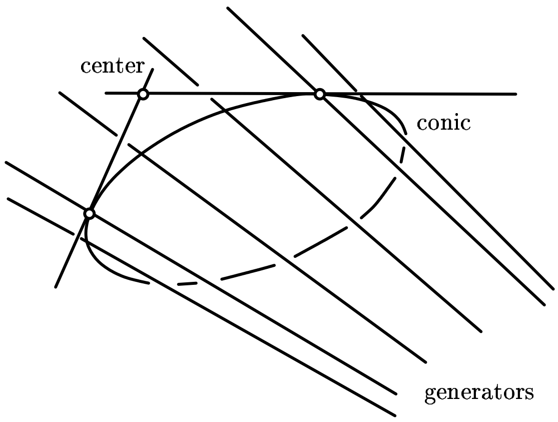

Fig. 9. Sketch of the situation in 4D.

Summing up, we have the following results.

Theorem 1. *Assume that the center $\mathbf{C} = (c_0, c_1, c_2, c_3, c_4)^\top$ of the projective mapping is not contained in the Klein image of the congruence, i.e., in $\mathbf{C} \notin M \cap T$. If $\mathbf{C}$ is contained in T, then the image of $\mathbf{L}(s,t)$ is a Cayley surface. If the coordinates of $\mathbf{C}$ satisfy $c_4^2 - c_2c_3 > 0$, then it is a Plücker conoid, otherwise it is a Zindler conoid.*

Proof. If one of the hyperplanes passes through the center $\mathbf{C}$, then it contains one tangent of the surface through $\mathbf{C}$, leading to a cuspidal point. Depending on the number of cuspidal points, we get the three different types of cubic ruled surfaces. □

point $\mathbf{p}$	$\longrightarrow$	line $\mathcal{L}_{\widehat{\mathcal{C}}}(\mathbf{p})$ through $\mathbf{p}$ which is contained in the congruence	$\longrightarrow$	Klein image (Plücker coordinates $\mathbf{L}_{\widehat{\mathcal{C}}}(\mathbf{p})$)	$\longrightarrow$	point $\pi(\mathbf{L}_{\widehat{\mathcal{C}}}(\mathbf{p}))$
$\Cap$		$\Cap$		$\Cap$		$\Cap$
3D space	$\longrightarrow$	line space (4D)	$\longrightarrow$	hyperquadric M in 5D space	$\longrightarrow$	cubic ruled surface $\pi(\widehat{\mathcal{C}})$ in 3D space

Fig. 10. Constructing a rational mapping onto a cubic ruled surface

Remark 7. If the center of the projective mapping is on the Klein image itself, then the image surface degenerates in various ways. For instance, one may obtain various types of ruled quadric surfaces. A more detailed discussion of these cases is beyond the scope of this article. Further results will be presented in [19].

4.3 Constructing curves and surfaces

In order to generate rational curves and surfaces on a given cubic ruled surface, one may now construct a suitable rational mapping, as follows. The mapping can be found by composing the mapping $\mathbf{p} \mapsto \mathbf{L}_{\widehat{\mathcal{C}}}(\mathbf{p})$ (see (27)), which maps each point in three–space to the Klein image of the unique congruence line through it, with a suitable projective mapping $\pi : P^5(\mathbb{R}) \rightarrow P^3(\mathbb{R})$, which maps the Klein image of the congruence into the desired cubic ruled surface. This process is summarized in Figure 10.

Since $\mathbf{L}_{\widehat{\mathcal{C}}}$ is a cubic rational mapping, the image of a curve of degree n is a curve of degree $3n$ on the cubic ruled surface. Similarly, the image of a tensor–product patch of degree (n, n) is a tensor–product patch of degree $(3n, 3n)$.

As an example, Figure 11 shows a bicubic patch on cubic ruled surface. In addition, four points are interpolated by a cubic curve. Two different examples (one of them with a point at infinity) are shown.

The cubic curve has been constructed by applying the rational mapping to a line $\mathcal{I}$ in three–dimensional space. The preimages of the four given points are four lines which belong to the congruence $\widehat{\mathcal{C}}$. The line $\mathcal{I}$ has to intersect these four preimage lines.

Generally, there may exist two lines which intersect all four lines. They can be found by intersecting the ruled quadric surface through the first three lines with the last one. This leads to two solutions, which may be conjugate–complex, or identical. In our situation, however, one of them is the focal line of the congruence, and the other line is the desired preimage of the cubic curve.

Following the ideas presented in [8, 9], the rational mapping can be used to construct rational curves and surfaces on the three types of cubic ruled surfaces.

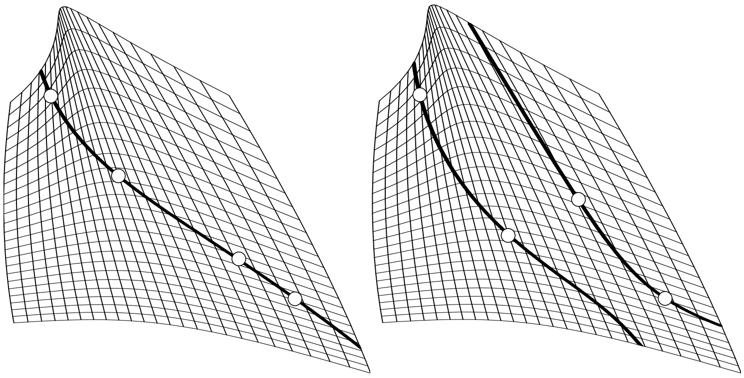

Fig. 11. Bicubic tensor-product patch on a cubic ruled surface. Interpolation of four points with a cubic (2 examples).

5 Line models of Veronese surfaces

Any line intersecting a space curve in two points is said to be a *chord* of it. We consider the line congruence $\widehat{\mathcal{V}}$ that consists of all chords of the twisted cubic curve

$$\mathcal{F}_1 = \mathcal{F}_2 = \{ \mathbf{p} \mid \mathbf{p} = (1, t, t^2, t^3)^\top,\ t \in \mathbb{C} \}. \tag{39}$$

Real lines are obtained either by connecting to real points on the curve, or by connecting two conjugate–complex ones. As already discussed in the second paragraph of section 4.2, the system of chords has the space–filling property: any point in three–dimensional space (except for the points on the cubic itself) belongs to exactly one chord. If the point is used as a center of a projective mapping into a plane, then corresponding two points on the cubic are mapped to the singular point of the resulting planar cubic.

As a well–known fact from advanced geometry, the system of chords is closely related to the Veronese surface [1].

Proposition 2. *The Klein image of the line congruence is a Veronese surface $\widehat{\mathcal{V}}$, which is contained in the hyperquadric M of five–dimensional real projective space.*

Proof. Recall that the Veronese surface in five–dimensional projective space is given by

$$(1, u, v, u^2, uv, v^2)^\top, \quad , u, v \in \mathbb{R}. \tag{40}$$

The Plücker coordinates of the chords connecting two points $(1, t, t^2, t^3)$ and $(1, s, s^2, s^3)$ of the twisted cubic can be shown to be equal to

$$(1, u, u^2 - v, v^2, -uv, v)^\top, \quad \text{with} \quad u = s + t, v = st, \tag{41}$$

where the common factor $(s - t)$ has been factored out. The Klein image of the chords is projectively equivalent to the Veronese surface (40). □

The Veronese surface can be seen as the generic triangular quadratic Bézier surface in five–dimensional space. By projecting it into three–dimensional space it is possible to obtain any triangular quadratic Bézier surface. This fact has been exploited in [5] in order to classify these surfaces. Similar to the discussion of the cubic ruled surfaces, the type of the result depends on the location of the (here one–dimensional) center of the mapping.

Proposition 3. *The Plücker coordinates of the line in $\widehat{\mathcal{V}}$ through a given point* $\mathbf{p}$ *are*

$$\begin{aligned}\mathbf{L}_{\widehat{\mathcal{V}}}(\mathbf{p}) = (\ & (p_1^2 - p_0\,p_2)^2, \quad (p_1^2 - p_0\,p_2)(p_1\,p_2 - p_0\,p_3), \\ & (p_0 p_2^3 + p_0^2 p_3^2 - 3\,p_0\,p_1\,p_2\,p_3 + p_1^3 p_3), \quad (p_2^2 - p_1\,p_3)^2, \\ & (p_1\,p_3 - p_2^2)(p_1\,p_2 - p_0\,p_3), \quad (p_1\,p_3 - p_2^2)(p_0\,p_2 - p_1^2) \ )^\top . \end{aligned} \tag{42}$$

Proof. This can be shown by computing the singular point of the planar cubic obtained by projecting the space cubic into the plane, where $\mathbf{p}$ serves as the center. The details are omitted. □

Similar to the ideas discussed in Section 4.3, the mapping

$$\mathbf{p} \mapsto \mathbf{L}_{\widehat{\mathcal{V}}}(\mathbf{p}) \tag{43}$$

can be used for parameterizing the images of the Veronese surface under projective mappings into three–dimensional space (which are all types of quadratic triangular Bézier surfaces).

6 Concluding remarks

Line congruences have been shown to be useful for the construction of rational curves and surfaces on special algebraic surfaces. In the case of quadric surfaces, this leads to an additional geometrical approach (which had already been outlined in [16]) to the generalized stereographic projection. Originally, this technique had earlier been derived mainly relying on algebraic results [8]. As shown in this paper, similar techniques are available for cubic ruled surfaces, and for Veronese surfaces (triangular Bézier surfaces).

As an obvious question, one may ask which line congruences provide the space–filling property, and – related to it – an associated rational mapping. This is related to the *bundle degree* (the number of lines passing through a generic points) of these congruences. The classification of line congruences with respect to their bundle degree has been studied mainly in the 19th century, in classical texts on algebraic line geometry. We mention the following results:

Generally, the lines connecting two different algebraic space curves of order m and n with s intersections (counted with multiplicities) form a congruence of bundle degree $mn - s$ [22]. Consequently, the construction of the line congruence $\widehat{\mathcal{C}}$ can immediately be generalized to congruences with two focal curves, where one of them is a straight line. For instance, the line congruence spanned by a space cubic and one of its chords (or tangents) has the space–filling property, since any plane through the chord (or tangent) intersects the cubic in exactly one additional point. Clearly, the Klein images of the congruences generated in this way are ruled surfaces. Other choices of m and n with $1 \notin \{m, n\}$ do not produce congruences with bundle degree 1, since the algebraic space cannot intersect in sufficiently many points without becoming identical.

The bundle degree b of the chords of an irreducible algebraic space curve of order n satisfies

$$\lfloor (n-1)^2/4 \rfloor \leq b \leq (n-1)(n-2)/2, \quad (44)$$

see [3, 23]. It depends on the types of singularities of the space curve. Only cubic curves ($n = 3$) give congruences with $b = 1$.

Note that there are other possibilities to define space–filling line congruences than the two possibilities described in this paper. For instance, one may take all lines which touch a given surface and pass through a curve. These lines form a congruence of bundle degree rn, where r is the rank of the surface (i.e., the algebraic order of its tangent cones) and n is the order of the space curve [22].

Further research will be devoted to possible generalizations of this approach, which may include systems of linear subspaces in spaces of dimension higher than three. In addition, we plan to develop computational techniques for generating rational curves and surfaces on these special algebraic surfaces, such as techniques for interpolation and approximation (see [6] for the quadric case). Also, we will analyze the relation to Müller's results on universal parameterizations of special cubic surfaces, which also cover the case of ruled ones [15]. Last but not least, we plan to complete the results on cubic ruled surfaces by analyzing the degenerate situations.

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